Essential obstacles to Helly circular-arc graphs

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Overview

Circular-arc graphs and Helly circular-arc graphs

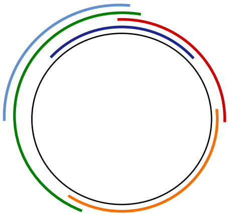
Obstacles

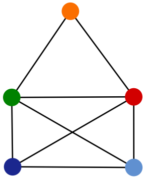
Essential obstacles

Our results

Circular-arc graphs

- The intersection graph of a set $\mathcal{A}$ of arcs on a circle is a graph having one vertex for each arc in $\mathcal{A}$ and such that two different vertices are adjacent if and only if the corresponding arcs have nonempty intersection.





- A graph G is a circular-arc graph (Tucker, 1971) if G is the intersection graph of some set $\mathcal{A}$ of arcs on a circle; if so, the set $\mathcal{A}$ is called a circular-arc model of G.

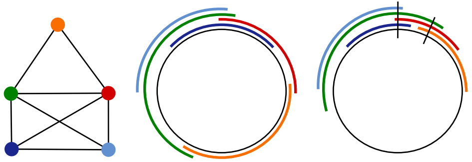
- The class of circular-arc graph is clearly closed by vertex removals and thus has some minimal forbidden induced subgraph characterization.

Forbidden structures and recognition algorithms

- Forbidden structures for the class of circular-arc graphs and its main subclasses, as well as efficient algorithms for finding such structures, have received a great deal of attention (Tucker, 1974; Trotter and Moore, 1976; Bang-Jensen and Hell, 1994; Feder et al., 1999; Hell and Huang, 2004; Lin et al., 2007; Lin and Szwarcfiter, 2008; Bonomo et al., 2009; Kaplan and Nussbaum, 2009; Joeris et al., 2011; Lin et al., 2013; Francis et al., 2014; Cao et al., 2015; Francis et al., 2015; Soulignac, 2015; Safe, 2016).
- A complete characterization by forbidden structures for the class of circular-arc graphs, together with an $O(n^3)$-time algorithm for finding one such forbidden structure in any given graph that is not a circular-arc graph, was recently given in (Francis et al., 2015).
- Linear-time recognition algorithms for circular-arc graphs were proposed in (McConnell, 2003; Kaplan and Nussbaum, 2006)

Helly circular-arc graphs

- A family of sets has the Helly property (Berge, 1973), or simply is Helly, if every nonempty subfamily of pairwise intersecting sets has nonempty total intersection.
- A Helly circular-arc graph (sometimes also Θ circular-arc graph) is a circular-arc graph admitting a Helly circular-arc model.



- These graphs were introduced by Gavril (1974).
- The class of Helly circular-arc graph is clearly closed by vertex removals and thus has some minimal forbidden induced subgraph characterization.

Recognition algorithms and forbidden structures

Gavril (1974) derived an $O(n^3)$-time recognition algorithm for them based on the circular-ones property for columns of their clique-matrices.

Joeris et al. (2011) gave a linear-time recognition algorithm for Helly circular-arc graphs. (The definition of obstacles is given later on).

Theorem (Joeris et al., 2011)

Given a circular-arc graph G, it is possible find in linear time either a Helly circular-arc model of G or an obstacle induced in G.
Moreover, if a circular-arc model of G is given as input, the time bound reduces to $O(n)$.

Joeris et al.'s algorithm is certifying, meaning that it produces an easy-to-check certificate for the correctness of its answer:

- If the input is a Helly circular-arc graph, their algorithm answers 'yes' together with a positive certificate, which consists of a Helly circular-arc model of the input graph.
- Otherwise, the answer is 'no' together with a negative certificate, which consists of an induced subgraph of the input graph that belongs to a family of graphs called obstacles (Joeris et al., 2011).

Obstacles

That an induced obstacle serves as a certificate of the 'no' answer follows from the structural result below.

Theorem (Joeris et al., 2011)

A circular-arc graph G is a Helly circular-arc graph if and only if G contains no induced obstacle.

The above theorem gives a characterization of Helly circular-arc graphs by forbidden induced subgraphs restricted to circular-arc graphs.

This characterization is not by minimal forbidden induced circular-arc subgraphs because:

- There are obstacle that contain obstacles with fewer vertices as induced subgraphs.
- THere are obstacles which are not circular-arc graphs.

Minimal circular-arc obstacles

Minimal circular-arc obstacles

We say an obstacle is minimal if it contains no induced obstacle having fewer vertices.
A minimal circular-arc obstacle is an obstacle that is both minimal and a circular-arc graph.

Joeris et al.'s characterization clearly holds replacing 'obstacle' by 'minimal circular-arc obstacle':

Theorem (Joeris et al., 2011)

A circular-arc graph G is a Helly circular-arc graph if and only if G contains no induced minimal circular-arc obstacle.

This is the characterization for the class of Helly circular-arc graphs by minimal forbidden induced subgraphs restricted to circular-arc graphs.

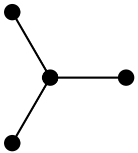
A partial list of minimal circular-arc obstacles was given in Bonomo et al. (2014).

Our contributions

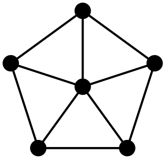
1. In this work, we introduce essential obstacles, a refinement of the notion of obstacles, and prove that essential obstacles are precisely the minimal circular-arc obstacles or, equivalently, the minimal forbidden induced circular-arc subgraphs for the class of Helly circular-arc graphs (where by a circular-arc subgraph we mean a subgraph which is a circular-arc graph).
2. Moreover, we show that, given any obstacle, it is possible to find in linear time a minimal forbidden induced subgraph for the class of Helly circular-arc graphs contained in it as an induced subgraph. Hence, given any negative certificate produced by Joeris et al.'s algorithm, it is possible to obtain a minimal negative certificate while preserving the linear time bound.

Our contributions

3. No characterization by forbidden induced subgraphs is known for Helly circular-arc graphs in the general case (i.e., not restricted to circular-arc graphs). As a partial result, we give the minimal forbidden induced subgraph characterization of Helly circular-arc graphs restricted to graphs containing no induced claw and no induced 5-wheel and show that it is possible to find in linear time, in any given graph that is not a Helly circular-arc graph, an induced subgraph isomorphic to claw, 5-wheel, or some minimal forbidden induced subgraph for the class of Helly circular-arc graphs.



claw



5-wheel

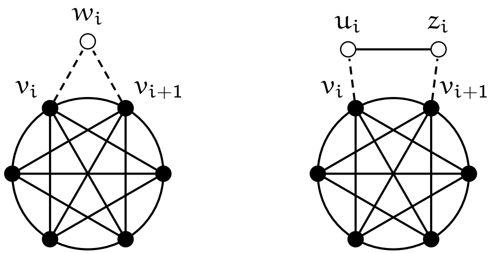
Obstacles

Obstacle enumerations and witnesses

An obstacle enumeration in a graph G is a circular enumeration $\mathcal{Q} = v_1, v_2, \ldots, v_k$ of the $k \geqslant 3$ vertices of a clique Q and, for each $i \in \{1, \ldots, k\}$, a linear enumeration $\mathcal{W}_i$ consisting of one or two vertices of G such that one of the following holds (subindices are modulo k):

($\mathcal{O}_1$) $\mathcal{W}_i = w_i$ where $N_{\overline{G}}(w_i) \cap Q = \{v_i, v_{i+1}\}$;

($\mathcal{O}_2$) $\mathcal{W}_i = u_i, z_i$ where $N_{\overline{G}}(u_i) \cap Q = \{v_i\}$, $N_{\overline{G}}(z_i) \cap Q = \{v_{i+1}\}$, and $u_i z_i \in E(G)$.

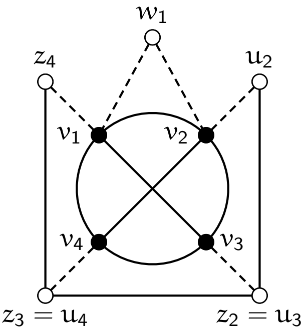


The vertices in the set $W(\mathcal{Q})$ formed by the vertices in the enumerations $\mathcal{W}_1, \ldots, \mathcal{W}_k$ are the witnesses of $\mathcal{Q}$.

Obstacles

Obstacles (Joeris et al., 2011)

An obstacle is a graph G admitting an obstacle enumeration $\mathcal{Q}$ such that $V(G) = V(\mathcal{Q}) \cup W(\mathcal{Q})$.



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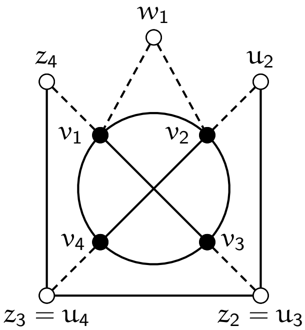

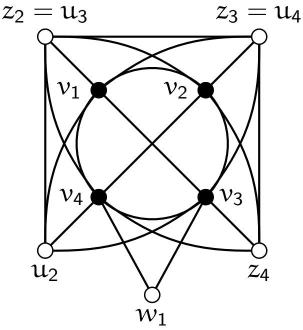

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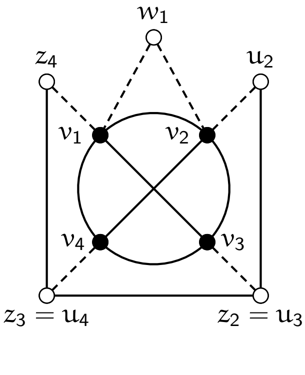

Except for those edges required by the definition of obstacle enumeration (u_2z_2, u_3z_3, and u_4z_4) adjacencies among white vertices are arbitrary.

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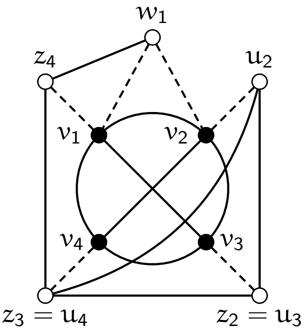

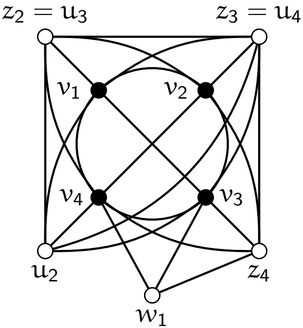

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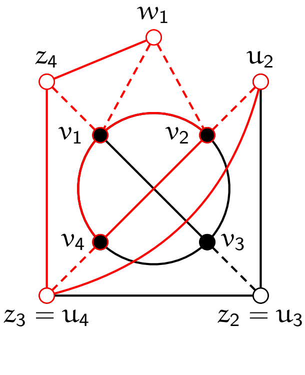
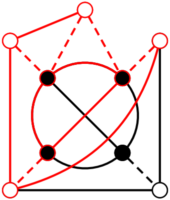
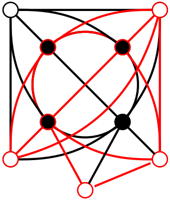


Except for those edges required by the definition of obstacle enumeration (u_2z_2, u_3z_3, and u_4z_4) adjacencies among white vertices are arbitrary.

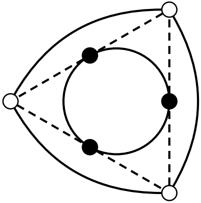
Obstacles may not be minimal or circular-arc

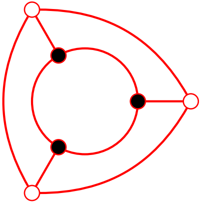
- Not only there are obstacles which contain other obstacles as induced subgraphs:





- As noticed already by Joeris et al. (2011), obstacles may not be circular-arc graphs.





$\overline{C_6}$ is not circular-arc

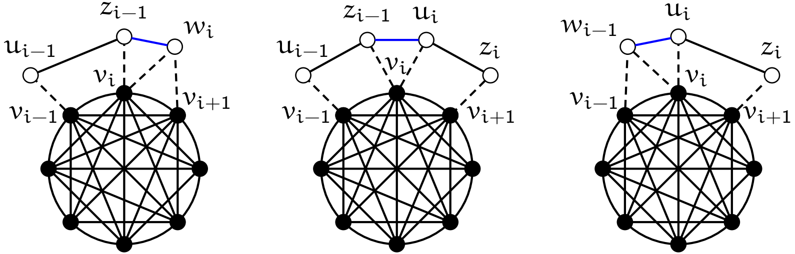
Valid edges

In order to introduce essential obstacles, we define valid edges.

Valid edges

Let $\mathcal{Q}$ be an obstacle enumeration in a graph G and let $Q = V(\mathcal{Q})$. Given an edge y_1y_2 of G joining two witnesses y_1 and y_2 of $\mathcal{Q}$, we say the edge y_1y_2 is valid if either $N_{\overline{G}}(y_1) \cap Q$ and $N_{\overline{G}}(y_2) \cap Q$ are comparable (i.e., one is a subset of the other) or y_1 and y_2 occur together in some witness enumeration of $\mathcal{Q}$.

Roughly speaking, if the witnesses of $\mathcal{Q}$ are labeled as in the definition of obstacles, then the edge y_1y_2 is valid if and only if y_1y_2 equals u_iz_i, $z_{i-1}w_i$, $z_{i-1}u_i$, or $w_{i-1}u_i$ for some $i \in \{1, \dots, k\}$.



Essential obstacles

We are ready to define essential enumerations and obstacles.

Essential enumerations

An obstacle enumeration $\mathcal{Q}$ is essential if every edge joining two of its witnesses is valid.

Essential obstacle

An obstacle is essential if there is an essential obstacle enumeration of it.

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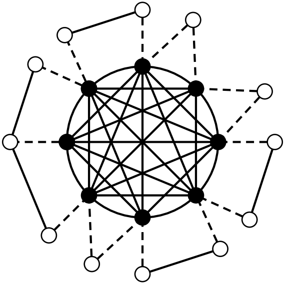

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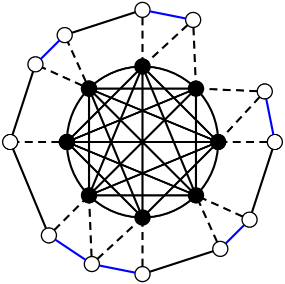

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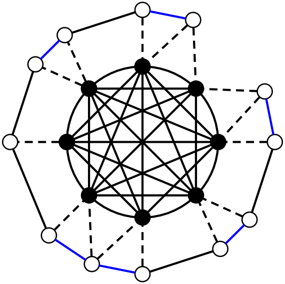

- Each of the blue edges may or may not be present.

Essential obstacles

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Essential enumerations

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Essential obstacle

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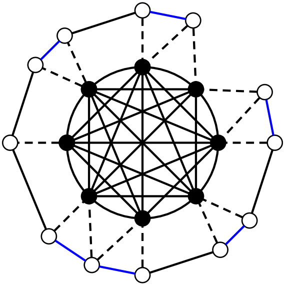


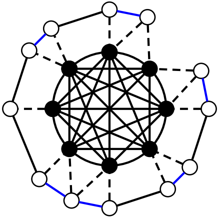
- Each of the blue edges may or may not be present.
- No other edges between white vertices are allowed.

Essential obstacles: Main results

Our main structural result is the the following.

Theorem

Essential obstacles are precisely the minimal forbidden induced circular-arc subgraphs for the class of Helly circular-arc graphs.



For the proof, we show that:

- each essential obstacle is a minimal circular-arc obstacle (i.e., is circular-arc and removing any single vertex leads to a Helly circular-arc graph); and
- each minimal circular-arc obstacle having some non-essential obstacle enumeration also has some essential obstacle enumeration and thus is an essential obstacle.

Essential obstacles: Main results

Our main algorithmic result is the following.

Theorem

Given a graph G and an obstacle enumeration $\mathcal{Q}$ in G, it is possible to find in linear time either an essential obstacle enumeration of some induced subgraph of G or an induced subgraph of G that is a minimal forbidden induced subgraph for the class of circular-arc graphs. Moreover, if G is a circular-arc graph, given a circular-arc model of G and an obstacle enumeration $\mathcal{Q}$ in G, an essential obstacle enumeration of some induced subgraph of G can be found in $O(n)$ time.

Together with Joeris et al. (2011), implies a linear-time recognition algorithm with minimal negative certificates if the input is circular-arc.

Corollary

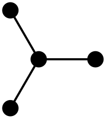
Given a circular-arc graph G, it is possible to find in linear time either a Helly circular-arc model of G or an essential obstacle enumeration of some minimal forbidden induced subgraph for the class of Helly circular-arc graphs contained in G as an induced subgraph. Moreover, if a circular-arc model of G is given as input, the time bound reduces to $O(n)$.

What about non-circular-arc graphs?

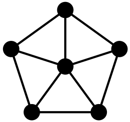
The absence of induced essential obstacles characterizes those circular-arc graphs G which are Helly circular-arc graphs.

No analogous forbidden induced subgraph characterization is known if G is not assumed to be a circular-arc graph.

Here, we give such a characterization restricted to $\{\text{claw}, \text{5-wheel}\}$-free graphs.



claw



5-wheel

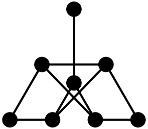
Notice that no forbidden induced subgraph characterization for circular-arc graphs restricted to $\{\text{claw}, \text{5-wheel}\}$-free graphs is known. It is known for the following more restricted classes: complements of bipartite graphs (Trotter and Moore, 1976) and to claw-free chordal graphs (Bonomo et al., 2009).

{claw, 5-wheel}-free Helly circular-arc graphs

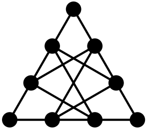
We find the minimal forbidden induced subgraph characterization of Helly circular-arc graphs restricted to {claw, 5-wheel}-free graphs.

Theorem

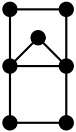
Let G be {claw, 5-wheel}-free graph. Then, G is a Helly circular-arc graph if and only if G contains no induced $\overline{3K_2}$, $\overline{P_7}$, $\overline{F_1}$, $\overline{F_2}$, $\overline{H_3}$, net, $\overline{2P_4}$, $\overline{F_8}$, $\overline{C_6}$, tent $+ K_1$, or $C_k + K_1$ for any $k \geqslant 4$.



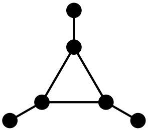
F_1



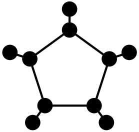
F_2



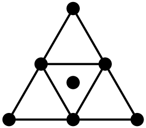
H_3



net



F_8



tent $+ K_1$

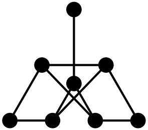
For the proof, we determine explicitly all claw-free essential obstacles and exploit a recent characterization for concave-round graphs (Safe, 2016).

{claw, 5-wheel}-free Helly circular-arc graphs

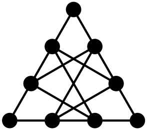
Moreover, we obtain a robust linear-time certifying recognition algorithm for Helly circular-arc graphs restricted to {claw, 5-wheel}-free graphs.

Theorem

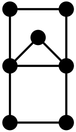
There is a linear-time algorithm that, given any graph G that is not a Helly circular-arc graph, finds an induced subgraph of G isomorphic to claw, 5-wheel, or one of the following minimal forbidden induced subgraphs for the class of Helly circular-arc graphs: $\overline{3K_2}$, $\overline{P_7}$, $\overline{F_1}$, $\overline{F_2}$, $\overline{H_3}$, net, $\overline{2P_4}$, $\overline{F_8}$, $\overline{C_6}$, tent $+ K_1$, or $C_k + K_1$ for any $k \geqslant 4$.



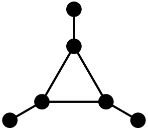
F_1



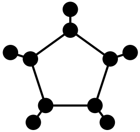
F_2



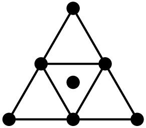
H_3



net



F_8



tent $+ K_1$

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Thank you very much for your attention!