Sobre la noción de conexión en espacios topológicos

Pablo Viola

XIV Congreso A. Monteiro, Bahía Blanca

02/junio/2017

Some notions of connection

A topological space (X, τ) is said:

(i) **(Standardly) Connected** if it cannot be written as the disjoint union of two open subsets.

Some notions of connection

A topological space (X, τ) is said:

(i) **(Standardly) Connected** if it cannot be written as the disjoint union of two open subsets.

(ii) **Path connected** if for every couple of points a, b, there is a continuous mapping ϕ from $[0, 1]$ into the space, such that $\phi(0) = a$ and $\phi(1) = b$.

Some notions of connection

A topological space (X, τ) is said:

(i) **(Standardly) Connected** if it cannot be written as the disjoint union of two open subsets.

(ii) **Path connected** if for every couple of points a, b, there is a continuous mapping ϕ from $[0, 1]$ into the space, such that $\phi(0) = a$ and $\phi(1) = b$.

(iii) **Hyperconnected** (or **irreducible**) if cannot be written as the union of two (disjoint or not) closed subsets.

Some notions of connection

A topological space (X, τ) is said:

(i) **(Standardly) Connected** if it cannot be written as the disjoint union of two open subsets.

(ii) **Path connected** if for every couple of points a, b, there is a continuous mapping ϕ from $[0, 1]$ into the space, such that $\phi(0) = a$ and $\phi(1) = b$.

(iii) **Hyperconnected** (or **irreducible**) if cannot be written as the union of two (disjoint or not) closed subsets.

(iv) **Ultraconnected** if no pair of closed sets are disjoint.

Some notions of connection

A topological space (X, τ) is said:

(i) **(Standardly) Connected** if it cannot be written as the disjoint union of two open subsets.

(ii) **Path connected** if for every couple of points a, b, there is a continuous mapping ϕ from $[0, 1]$ into the space, such that $\phi(0) = a$ and $\phi(1) = b$.

(iii) **Hyperconnected** (or **irreducible**) if cannot be written as the union of two (disjoint or not) closed subsets.

(iv) **Ultraconnected** if no pair of closed sets are disjoint.

(v) **Simply connected** if it is path-connected, and whenever $p : [0, 1] \to X$ and $q : [0, 1] \to X$ are continuous maps such that $p(0) = q(0)$ and $p(1) = q(1)$, then p and q are homotopic relative to $\{0, 1\}$.

What about an abstract notion of connection?

- A connection notion on a topological space X is well stablished for every subspace.

What about an abstract notion of connection?

- A connection notion on a topological space X is well stablished for every subspace.
- In a given subspace A of X, the connection notion becomes an equivalence relation $\sim_A$ on A.
  The equivalence classes are the **components** of the connection notion in that particular set A.

What about an abstract notion of connection?

- A connection notion on a topological space X is well stablished for every subspace.
- In a given subspace A of X, the connection notion becomes an equivalence relation $\sim_A$ on A.
  The equivalence classes are the **components** of the connection notion in that particular set A.
- If $A \subset B \subset X$, then every component in A is a component in B.

What about an abstract notion of connection?

- A connection notion on a topological space X is well stablished for every subspace.
- In a given subspace A of X, the connection notion becomes an equivalence relation $\sim_A$ on A.
  The equivalence classes are the **components** of the connection notion in that particular set A.
- If $A \subset B \subset X$, then every component in A is a component in B.
- A desirable property would be that the connection notion be preserved under continuous mappings, as far as possible.

Trying an axiomatic approach

Here we give a very broad and abstract notion of a connection property, by means of a list of Axioms:

Trying an axiomatic approach

Here we give a very broad and abstract notion of a connection property, by means of a list of Axioms:

Given a topological space (X, τ) we say that κ is a **connection on** X if κ is a family $\kappa = \{(A, \sim_A) | A \subset X\}$ such that:

A1 Each $\sim_A \subset A \times A$ is an equivalence relation on A. An $\sim_A$-equivalence class is called a κ-**component of** A.

Trying an axiomatic approach

Here we give a very broad and abstract notion of a connection property, by means of a list of Axioms:

Given a topological space (X, τ) we say that κ is a **connection on** X if κ is a family $\kappa = \{(A, \sim_A) | A \subset X\}$ such that:

- A1 Each $\sim_A \subset A \times A$ is an equivalence relation on A. An $\sim_A$-equivalence class is called a κ-**component of** A.
- A2 If $A \subset B$, then every κ-component of A is contained in some κ-component of B.

Trying an axiomatic approach

Here we give a very broad and abstract notion of a connection property, by means of a list of Axioms:

Given a topological space (X, τ) we say that κ is a **connection on** X if κ is a family $\kappa = \{(A, \sim_A) | A \subset X\}$ such that:

- A1 Each $\sim_A \subset A \times A$ is an equivalence relation on A. An $\sim_A$-equivalence class is called a κ-**component of** A.
- A2 If $A \subset B$, then every κ-component of A is contained in some κ-component of B.
- A3 If A, B are subspaces of X, then, for all $a, b \in A$, $a \sim_A b$ implies $f(a) \sim_{A \cup B} f(b)$.

Trying an axiomatic approach

Here we give a very broad and abstract notion of a connection property, by means of a list of Axioms:

Given a topological space (X, τ) we say that κ is a **connection on** X if κ is a family $\kappa = \{(A, \sim_A) | A \subset X\}$ such that:

- A1 Each $\sim_A \subset A \times A$ is an equivalence relation on A. An $\sim_A$-equivalence class is called a κ-**component of** A.
- A2 If $A \subset B$, then every κ-component of A is contained in some κ-component of B.
- A3 If A, B are subspaces of X, then, for all $a, b \in A$, $a \sim_A b$ implies $f(a) \sim_{A \cup B} f(b)$.

Trying an axiomatic approach

Here we give a very broad and abstract notion of a connection property, by means of a list of Axioms:

Given a topological space (X, τ) we say that κ is a **connection on** X if κ is a family $\kappa = \{(A, \sim_A) | A \subset X\}$ such that:

- A1 Each $\sim_A \subset A \times A$ is an equivalence relation on A. An $\sim_A$-equivalence class is called a κ-**component of** A.
- A2 If $A \subset B$, then every κ-component of A is contained in some κ-component of B.
- A3 If A, B are subspaces of X, then, for all $a, b \in A$, $a \sim_A b$ implies $f(a) \sim_{A \cup B} f(b)$.

We say that κ is a **tight connection on** X if, in addition, satisfies

- A4 If A, B, are subspaces of X, then every continuous function $f : A \to B$ preserves κ, that is, if $a, b \in A$ and $a \sim_A b$ then $f(a) \sim_B f(b)$.

Continuous image preservation

Standard connection, path connection, hyperconnection and ultraconnection are preserved by continuous mappings, but simply connection is not.

Continuous image preservation

Standard connection, path connection, hyperconnection and ultraconnection are preserved by continuous mappings, but simply connection is not.

In this talk, we are concerned with connection properties which are preserved by continuous mappings.

Basic examples

Every topological space has always the following tight connections:

Basic examples

Every topological space has always the following tight connections:

- **Indiscrete (or minimal) connection.**

$$\Gamma = \{(A, \sim_A) | A \subset X, \sim_A = A \times A\}.$$

- **Discrete (or maximal) connection.** On every A, the equivalence relation Δ_A is taken as the equality relation, so that each equivalence class has exactly one point. Thus, $\Delta_A = \{(a, a) | a \in A\}$. We will denote

$$\Delta = \{(A, \Delta_A) | A \subset X\}\}.$$

- **Standard connection.** On every A, the equivalence relation σ_A is defined as $a \sigma_A b$ if given two open sets U, V in A (in the subspace topology τ_A) such that $a \in U$, $b \in V$, and $A = U \cup V$, then we always have $U \cap V \neq \emptyset$. We will denote

$$\Sigma = \{(A, \sigma_A) | A \subset X\}.$$

- **Path connection.** On every A, the equivalence relation χ_A is defined by: $a, b \in A$, $a \chi_A b$ if there is a continuous function $f : [0, 1] \to A$ such that $f(0) = a$, $f(1) = b$. We will denote

$$\Pi = \{(A, \chi_A) | A \subset X\}.$$

The partial order of connections

Given a topological space (X, τ) and two connections $\kappa = \{(A, \sim_A) | A \subset X\}$, $\lambda = \{(A, \approx_A) | A \subset X\}$ on X, we say that κ **is finer than** λ, or that λ **is coarser than** κ, and we write $\kappa \sqsubset \lambda$ or $\lambda \sqsupset \kappa$, if for every $A \subset X$ every κ-component on A is contained in some λ-component on A.

The partial order of connections

Given a topological space (X, τ) and two connections $\kappa = \{(A, \sim_A) | A \subset X\}$, $\lambda = \{(A, \approx_A) | A \subset X\}$ on X, we say that κ **is finer than** λ, or that λ **is coarser than** κ, and we write $\kappa \sqsubset \lambda$ or $\lambda \sqsupset \kappa$, if for every $A \subset X$ every κ-component on A is contained in some λ-component on A.

Theorem.

Let (X, τ) be a topological space. Then $\sqsubset$ is a partial order on the family of all connections on (X, τ).

The partial order of connections

Given a topological space (X, τ) and two connections $\kappa = \{(A, \sim_A) | A \subset X\}$, $\lambda = \{(A, \approx_A) | A \subset X\}$ on X, we say that κ **is finer than** λ, or that λ **is coarser than** κ, and we write $\kappa \sqsubset \lambda$ or $\lambda \sqsupset \kappa$, if for every $A \subset X$ every κ-component on A is contained in some λ-component on A.

Theorem.

Let (X, τ) be a topological space. Then $\sqsubset$ is a partial order on the family of all connections on (X, τ).

Also, we make the following general observation:

$$\Delta \sqsubset \kappa \sqsubset \Gamma, \qquad \text{all connection } \kappa \text{ on } (X, \tau).$$

The lattice of connections

Theorem.

Given a topological space (X, τ) and a family $\mathcal{C} = \{\kappa_j | j \in J\}$ of connections in (X, τ), $\kappa_j = \{\sim_{j,A} | A \subset X\}$, the following family

$$\bigwedge \mathcal{C} = \Big\{ \big(A, \bigcap \{\sim_{j,A} | j \in J\}\big) | A \subset X \Big\}.$$

is a connection on (X, τ).

The lattice of connections

Theorem.

Given a topological space (X, τ) and a family $\mathcal{C} = \{\kappa_j | j \in J\}$ of connections in (X, τ), $\kappa_j = \{\sim_{j,A} | A \subset X\}$, the following family

$$\bigwedge \mathcal{C} = \Big\{ \big(A, \bigcap \{\sim_{j,A} | j \in J\}\big) | A \subset X \Big\}.$$

is a connection on (X, τ).

Definition.

We call $\bigwedge \mathcal{C}$ the **infimum connection of $\mathcal{C}$**.

The lattice of connections

Theorem.

Given a topological space (X, τ) and a family $\mathcal{C} = \{\kappa_j | j \in J\}$ of connections in (X, τ), $\kappa_j = \{\sim_{j,A} | A \subset X\}$, the following family

$$\bigwedge \mathcal{C} = \Big\{ \big(A, \bigcap \{\sim_{j,A} | j \in J\}\big) | A \subset X \Big\}.$$

is a connection on (X, τ).

Definition.

We call $\bigwedge \mathcal{C}$ the **infimum connection of $\mathcal{C}$**.
We call **supremum connection of $\mathcal{C}$** to

$$\bigvee \mathcal{C} = \bigwedge \{\lambda | \lambda \sqsupset \kappa_j, j \in J\}.$$

It is well defined since any connection is finer than Γ.

The lattice of connections

Theorem.

Given a topological space (X,τ) and a family $\mathcal{C} = \{\kappa_j | j \in J\}$ of connections in (X,τ), $\kappa_j = \{\sim_{j,A} | A \subset X\}$, the following family

$$\bigwedge \mathcal{C} = \Big\{ \big(A, \bigcap \{\sim_{j,A} | j \in J\}\big) | A \subset X \Big\}.$$

is a connection on (X,τ).

Definition.

We call $\bigwedge \mathcal{C}$ the **infimum connection of $\mathcal{C}$**.
We call **supremum connection of $\mathcal{C}$** to

$$\bigvee \mathcal{C} = \bigwedge \{\lambda | \lambda \sqsupset \kappa_j, j \in J\}.$$

It is well defined since any connection is finer than Γ.
Given two connections κ, λ, we denote:

$$\kappa \wedge \lambda = \bigwedge \{\kappa, \lambda\}, \qquad \kappa \vee \lambda = \bigvee \{\kappa, \lambda\}.$$

The lattice of connections

Theorem.

Given a topological space (X, τ) and a family $\mathcal{C} = \{\kappa_j | j \in J\}$ of connections in (X, τ), $\kappa_j = \{\sim_{j,A} | A \subset X\}$, the following family

$$\bigwedge \mathcal{C} = \Big\{ \big(A, \bigcap \{\sim_{j,A} | j \in J\}\big) | A \subset X \Big\}.$$

is a connection on (X, τ).

Definition.

We call $\bigwedge \mathcal{C}$ the **infimum connection of** $\mathcal{C}$.
We call **supremum connection of** $\mathcal{C}$ to

$$\bigvee \mathcal{C} = \bigwedge \{\lambda | \lambda \sqsupset \kappa_j, j \in J\}.$$

It is well defined since any connection is finer than Γ.
Given two connections κ, λ, we denote:

$$\kappa \wedge \lambda = \bigwedge \{\kappa, \lambda\}, \qquad \kappa \vee \lambda = \bigvee \{\kappa, \lambda\}.$$

We can talk about the **connection lattice** $\mathcal{L}_\tau$ over (X, τ).

Extended connections

Subspace connection.

Given the connection lattice $\kappa = \{(A, \sim_A) | A \subset X\}$ of a topological space (X, τ), if S is a subspace of X, then $\kappa|_S = \{(A, \sim_A) | A \subset S\}$ is a connection on S with the subspace topology.

Extended connections

Subspace connection.

Given the connection lattice $\kappa = \{(A, \sim_A) | A \subset X\}$ of a topological space (X, τ), if S is a subspace of X, then $\kappa|_S = \{(A, \sim_A) | A \subset S\}$ is a connection on S with the subspace topology.

Extended connection.

Given a topological space (X, τ) and a connection $\mu = \{(A, \sim_A) | A \subset S\}$ on some subspace S of X. If κ is a connection on X such that $\kappa|_S = \mu$, then we say that κ is an **extension** of μ.

Building extensions.

Let us define $\breve{\mu} = \{(A, \approx_A) | A \subset X\}$ in such a way that:

$$\approx_A = \begin{cases} \sim_A, & A \subset S \\ A \times A, & A \not\subset S. \end{cases}$$

It is not hard to see that $\breve{\mu}$ is an extension of μ on X, and it is called the **maximal extension** of μ.

Building extensions.

Let us define $\check{\mu} = \{(A, \approx_A) | A \subset X\}$ in such a way that:

$$\approx_A = \begin{cases} \sim_A, & A \subset S \\ A \times A, & A \not\subset S. \end{cases}$$

It is not hard to see that $\check{\mu}$ is an extension of μ on X, and it is called the **maximal extension** of μ.

Now, it is possible to define an extension $\hat{\mu}$ of μ on X, called the **minimal extension**, as

$$\hat{\mu} = \bigwedge \{\kappa | \kappa \text{ is an extension of } \mu \text{ on } X\}.$$

Building extensions.

Let us define $\check{\mu} = \{(A, \approx_A) | A \subset X\}$ in such a way that:

$$\approx_A = \begin{cases} \sim_A, & A \subset S \\ A \times A, & A \not\subset S. \end{cases}$$

It is not hard to see that $\check{\mu}$ is an extension of μ on X, and it is called the **maximal extension** of μ.

Now, it is possible to define an extension $\hat{\mu}$ of μ on X, called the **minimal extension**, as

$$\hat{\mu} = \bigwedge \{\kappa | \kappa \text{ is an extension of } \mu \text{ on } X\}.$$

It can be easily seen that, for any other extension κ of μ on X:

$$\hat{\mu} \sqsubset \kappa \sqsubset \check{\mu}.$$

Basic sublattice

Given a topological space (X, τ), and a set $E \subset X$, we call **trivial connection based on** E to the connection $\check{\Delta}^E$, where Δ^E is the discrete connection on E. If E is a single point, it is easily checked that $\check{\Delta}^E$ is the indiscrete connection on X. Hence, the family

$$\mathcal{L}_\tau^0 = \{\check{\Delta}^E | E \subset X\}$$

is a lattice contained in the connection lattice $\mathcal{L}_\tau$, which is closed under the $\wedge$ and $\vee$ operations. We say that the connection lattice $\mathcal{L}_\tau$ is **trivial** if $\mathcal{L}_\tau = \mathcal{L}_\tau^0$.

Basic sublattice

Given a topological space (X, τ), and a set $E \subset X$, we call **trivial connection based on** E to the connection $\check{\Delta}^E$, where Δ^E is the discrete connection on E. If E is a single point, it is easily checked that $\check{\Delta}^E$ is the indiscrete connection on X. Hence, the family

$$\mathcal{L}_\tau^0 = \{\check{\Delta}^E | E \subset X\}$$

is a lattice contained in the connection lattice $\mathcal{L}_\tau$, which is closed under the $\wedge$ and $\vee$ operations. We say that the connection lattice $\mathcal{L}_\tau$ is **trivial** if $\mathcal{L}_\tau = \mathcal{L}_\tau^0$.

Connection lattice for the discrete topology

For the discrete topology, the connection lattice is trivial.

Fit extension of a connection on a subspace

Consider a topological space (X, τ), a subspace $E \subset X$, and a connection κ on E. We define the **fit extension of the connection κ to the whole** X, as the finnest connection $\hat{\kappa}$ on X such that its restriction to E is coarser than or equal to κ. In symbols:

$$\hat{\kappa} = \bigwedge \{\lambda \in \mathcal{L}_\tau | \lambda|_E \sqsupset \kappa\}.$$

Since $\check{\kappa}$ is a connection on X which extends κ, the fit connection $\hat{\kappa}$ is well defined.

Finest product connection

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces and denote $Y = \prod\{X_j | j \in J\}$, its standard product topology.

Finest product connection

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces and denote $Y = \prod\{X_j | j \in J\}$, its standard product topology.

Consider a family $\{\kappa_j | j \in J\}$, where each κ_j is a connection on X_j.

Finest product connection

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces and denote $Y = \prod\{X_j | j \in J\}$, its standard product topology.

Consider a family $\{\kappa_j | j \in J\}$, where each κ_j is a connection on X_j.

Given a point $\vec{o} = (o_j)_{j \in J} \in Y$ and $k \in J$, the **translated spaces** are

$$X_k^{\|\vec{o}} = \prod_{j \in J} \begin{cases} \{o_j\}, & j \neq k \\ X_k, & j = k. \end{cases}$$

Finest product connection

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces and denote $Y = \prod\{X_j | j \in J\}$, its standard product topology.

Consider a family $\{\kappa_j | j \in J\}$, where each κ_j is a connection on X_j.

Given a point $\vec{o} = (o_j)_{j \in J} \in Y$ and $k \in J$, the **translated spaces** are

$$X_k^{\|\vec{o}} = \prod_{j \in J} \begin{cases} \{o_j\}, & j \neq k \\ X_k, & j = k. \end{cases}$$

For each $\vec{o} \in Y$ and every $k \in J$, the translated space $X_j^{\|\vec{o}}$ is a subspace of Y, obviously homeomorphic to X_k.

Finest product connection

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces and denote $Y = \prod\{X_j | j \in J\}$, its standard product topology.

Consider a family $\{\kappa_j | j \in J\}$, where each κ_j is a connection on X_j.

Given a point $\vec{o} = (o_j)_{j \in J} \in Y$ and $k \in J$, the **translated spaces** are

$$X_k^{\|\vec{o}} = \prod_{j \in J} \begin{cases} \{o_j\}, & j \neq k \\ X_k, & j = k. \end{cases}$$

For each $\vec{o} \in Y$ and every $k \in J$, the translated space $X_j^{\|\vec{o}}$ is a subspace of Y, obviously homeomorphic to X_k.
In an analogous manner, we can talk about the **translated connection** $\kappa_j^{\|\vec{o}}$, which is transported by the standard homeomorphism $X_j \to X_j^{\|\vec{o}}$.

Finest product connection

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces and denote $Y = \prod\{X_j | j \in J\}$, its standard product topology.

Consider a family $\{\kappa_j | j \in J\}$, where each κ_j is a connection on X_j.

Given a point $\vec{o} = (o_j)_{j \in J} \in Y$ and $k \in J$, the **translated spaces** are

$$X_k^{\|\vec{o}} = \prod_{j \in J} \begin{cases} \{o_j\}, & j \neq k \\ X_k, & j = k. \end{cases}$$

For each $\vec{o} \in Y$ and every $k \in J$, the translated space $X_j^{\|\vec{o}}$ is a subspace of Y, obviously homeomorphic to X_k.
In an analogous manner, we can talk about the **translated connection** $\kappa_j^{\|\vec{o}}$, which is transported by the standard homeomorphism $X_j \to X_j^{\|\vec{o}}$.

The **finest product connection** $\curlywedge\{\kappa_j | j \in J\}$ on the product space Y is

$$\kappa^{\curlywedge} = \bigcurlywedge\{\kappa_j | j \in J\} = \bigvee\left\{\hat{\kappa}_j^{\|\vec{o}} \,\middle|\, j \in J, \vec{o} \in Y\right\}.$$

Product connections

Now, given a subspace $A \subset Y$, we define $A^{\square}$ as the set

$$A^{\square} = \prod \{\pi_j(A) | j \in J\},$$

where $\pi_j : Z \to X_j$ is the projection of the j-th coordinate, each $j \in J$.

Product connections

Now, given a subspace $A \subset Y$, we define $A^{\Box}$ as the set

$$A^{\Box} = \prod \{\pi_j(A) | j \in J\},$$

where $\pi_j : Z \to X_j$ is the projection of the j-th coordinate, each $j \in J$.

For the finest product connection, let us denote $\kappa^{\curlywedge} = \{(A, \sim_A) | A \subset Y\}$.

Product connections

Now, given a subspace $A \subset Y$, we define $A^{\Box}$ as the set

$$A^{\Box} = \prod\{\pi_j(A) | j \in J\},$$

where $\pi_j : Z \to X_j$ is the projection of the j-th coordinate, each $j \in J$.

For the finest product connection, let us denote $\kappa^{\curlywedge} = \{(A, \sim_A) | A \subset Y\}$. Given a set $A \subset Z$ and two points $\vec{a} = (a_j)_{j\in J}$, $\vec{b} = (b_j)_{j\in J}$, in A, we define

$$a \approx_A b \quad \Leftrightarrow \quad a \sim_{A^{\Box}} b.$$

It is not hard to proof that

$$\kappa^{\curlyvee} = \bigcurlyvee\{\kappa_j | j \in J\} = \{(A, \approx_A) | A \subset Z\}$$

is a connection on Z, and we call it **coarser product connection**.

Product connections

Now, given a subspace $A \subset Y$, we define $A^{\square}$ as the set

$$A^{\square} = \prod \{\pi_j(A) | j \in J\},$$

where $\pi_j : Z \to X_j$ is the projection of the j-th coordinate, each $j \in J$.

For the finest product connection, let us denote $\kappa^{\curlywedge} = \{(A, \sim_A) | A \subset Y\}$. Given a set $A \subset Z$ and two points $\vec{a} = (a_j)_{j \in J}$, $\vec{b} = (b_j)_{j \in J}$, in A, we define

$$a \approx_A b \quad \Leftrightarrow \quad a \sim_{A^{\square}} b.$$

It is not hard to proof that

$$\kappa^{\curlyvee} = \curlyvee \{\kappa_j | j \in J\} = \{(A, \approx_A) | A \subset Z\}$$

is a connection on Z, and we call it **coarser product connection**.

We say that a connection κ on Z is a **product connection** if

$$\kappa^{\curlywedge} \sqsubset \kappa \sqsubset \kappa^{\curlyvee}.$$

Proposition.

Let $\{(X_j, \tau_j) | j \in J\}$ be a family of topological spaces, and for each $j \in J$ denote Σ_j to the standard connection on X_j. Then the coarser connection product $\curlyvee \{\Sigma_j | j \in J\}$ is the standard connection Σ on the product space $Y = \prod\{X_j | j \in J\}$.

Structure of the connection lattice

Proposition

Let (X, τ) be a topological space such that for every pair of non-discrete connections on X, their infimum is also non-discrete.
Let κ be a non-discrete connection such that X has at least one κ-component having three elements or more. Then there is another non-discrete connection λ which is strictly finer.

Structure of the connection lattice

Proposition

Let (X, τ) be a topological space such that for every pair of non-discrete connections on X, their infimum is also non-discrete.
Let κ be a non-discrete connection such that X has at least one κ-component having three elements or more. Then there is another non-discrete connection λ which is strictly finer.

Theorem.

Let (X, τ) a topological space such that for every pair of non-discrete connections on X, their infimum is also non-discrete.
Then, for every non-discrete connection κ on X having at least one infinite κ-component C on X, there is another non-discrete connection λ on X such that λ is strictly finer than κ, also having at least one infinite λ-component D on X.

Structure of the connection lattice

Proposition

Let (X, τ) be a topological space such that for every pair of non-discrete connections on X, their infimum is also non-discrete.
Let κ be a non-discrete connection such that X has at least one κ-component having three elements or more. Then there is another non-discrete connection λ which is strictly finer.

Theorem.

Let (X, τ) a topological space such that for every pair of non-discrete connections on X, their infimum is also non-discrete.
Then, for every non-discrete connection κ on X having at least one infinite κ-component C on X, there is another non-discrete connection λ on X such that λ is strictly finer than κ, also having at least one infinite λ-component D on X.

Definition

We say that a connection lattice $\mathcal{L}_\kappa$ is reach if satisfies the hypothesis of the Theorem above.

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

The lattice $\mathcal{L}_\tau^*$ is closed by the $\wedge$ and $\vee$, so it is a sublattice of $\mathcal{L}_\tau$. Its named the **reach connection sublattice** of $\mathcal{L}_\tau$.

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

The lattice $\mathcal{L}_\tau^*$ is closed by the $\wedge$ and $\vee$, so it is a sublattice of $\mathcal{L}_\tau$. Its named the **reach connection sublattice** of $\mathcal{L}_\tau$.

The indiscrete connection Γ belongs to $\mathcal{L}_\tau^*$, but the discrete connection does not.

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

The lattice $\mathcal{L}_\tau^*$ is closed by the $\wedge$ and $\vee$, so it is a sublattice of $\mathcal{L}_\tau$. Its named the **reach connection sublattice** of $\mathcal{L}_\tau$.

The indiscrete connection Γ belongs to $\mathcal{L}_\tau^*$, but the discrete connection does not.

If $\kappa \in \mathcal{L}_\tau^*$, then there exists $\lambda \in \mathcal{L}_\tau^*$ such that

$$\lambda \sqsubset \kappa, \qquad \lambda \neq \kappa.$$

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

The lattice $\mathcal{L}_\tau^*$ is closed by the $\wedge$ and $\vee$, so it is a sublattice of $\mathcal{L}_\tau$. Its named the **reach connection sublattice** of $\mathcal{L}_\tau$.

The indiscrete connection Γ belongs to $\mathcal{L}_\tau^*$, but the discrete connection does not.

If $\kappa \in \mathcal{L}_\tau^*$, then there exists $\lambda \in \mathcal{L}_\tau^*$ such that

$$\lambda \sqsubset \kappa, \qquad \lambda \neq \kappa.$$

The real numbers

- If κ is a non-discrete connection of $\mathbb{R}$, then $\mathbb{R}$ is a κ-component on itself.

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

The lattice $\mathcal{L}_\tau^*$ is closed by the $\wedge$ and $\vee$, so it is a sublattice of $\mathcal{L}_\tau$. Its named the **reach connection sublattice** of $\mathcal{L}_\tau$.

The indiscrete connection Γ belongs to $\mathcal{L}_\tau^*$, but the discrete connection does not.

If $\kappa \in \mathcal{L}_\tau^*$, then there exists $\lambda \in \mathcal{L}_\tau^*$ such that

$$\lambda \sqsubset \kappa, \qquad \lambda \neq \kappa.$$

The real numbers

- If κ is a non-discrete connection of $\mathbb{R}$, then $\mathbb{R}$ is a κ-component on itself.
- This implies that every pair of non-discrete connections on $\mathbb{R}$ have a non-discrete infimum.

Reach connection sublattice

Given a reach connection lattice $\mathcal{L}_\tau$, we denote $\mathcal{L}_\tau^*$ the collection of all connections $\kappa \in \mathcal{L}_\tau$ having some infinite component.

The lattice $\mathcal{L}_\tau^*$ is closed by the $\wedge$ and $\vee$, so it is a sublattice of $\mathcal{L}_\tau$. Its named the **reach connection sublattice** of $\mathcal{L}_\tau$.

The indiscrete connection Γ belongs to $\mathcal{L}_\tau^*$, but the discrete connection does not.

If $\kappa \in \mathcal{L}_\tau^*$, then there exists $\lambda \in \mathcal{L}_\tau^*$ such that

$$\lambda \sqsubset \kappa, \qquad \lambda \neq \kappa.$$

The real numbers

- If κ is a non-discrete connection of $\mathbb{R}$, then $\mathbb{R}$ is a κ-component on itself.
- This implies that every pair of non-discrete connections on $\mathbb{R}$ have a non-discrete infimum.
- The connection lattice of $\mathbb{R}$ is reach.

TO BE CONTINUED...