Independence number of products of Kneser graphs

Mario Valencia-Pabon

LIPN, Université Paris-Nord, France.

(Joint-work with Boštjan Brešar)

SODyG 2018 UNS, Argentina, 11/12/2018

Preliminaries

- $G = (V, E)$, where V is the vertex-set and E is the edge-set.
- A set $S \subseteq V$ is independent (or stable) in G if any $u, v \in S$ are non-adjacent in G.
- $\alpha(G) =$ maximum cardinality of an independent set in G.

Preliminaries

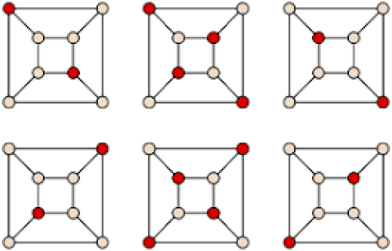
- $G = (V, E)$, where V is the vertex-set and E is the edge-set.
- A set $S \subseteq V$ is independent (or stable) in G if any $u, v \in S$ are non-adjacent in G.
- $\alpha(G) =$ maximum cardinality of an independent set in G.

Preliminaries

- $G = (V, E)$, where V is the vertex-set and E is the edge-set.
- A set $S \subseteq V$ is independent (or stable) in G if any $u, v \in S$ are non-adjacent in G.
- $\alpha(G) =$ maximum cardinality of an independent set in G.

Preliminaries

- $G = (V, E)$, where V is the vertex-set and E is the edge-set.
- A set $S \subseteq V$ is independent (or stable) in G if any $u, v \in S$ are non-adjacent in G.
- $\alpha(G) =$ maximum cardinality of an independent set in G.



Most common products of graphs

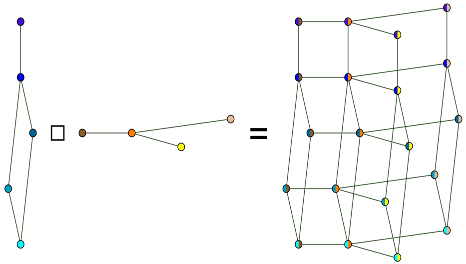
- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *cartesian product* $G \square H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $(g_1 g_2 \in E(G)$ and $h_1 = h_2)$ or $(g_1 = g_2$ and $h_1 h_2 \in E(H))$.

Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *cartesian product* $G \Box H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $(g_1 g_2 \in E(G)$ and $h_1 = h_2)$ or $(g_1 = g_2$ and $h_1 h_2 \in E(H))$.

Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *cartesian product* $G \square H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $(g_1 g_2 \in E(G)$ and $h_1 = h_2)$ or $(g_1 = g_2$ and $h_1 h_2 \in E(H))$.



Most common products of graphs

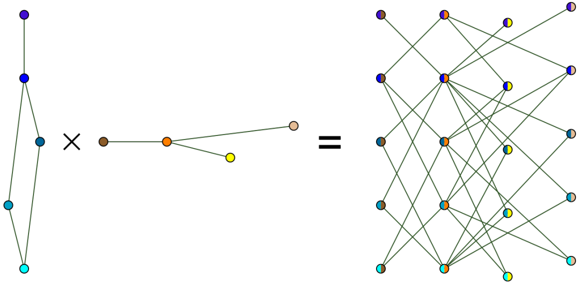
- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *direct product* $G \times H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $(g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H))$.

Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *direct product* $G \times H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $(g_1g_2 \in E(G) \text{ and } h_1h_2 \in E(H))$.

Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *direct product* $G \times H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $(g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H))$.



Most common products of graphs

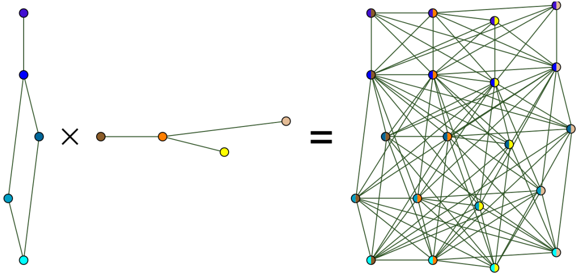
- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *strong product* $G \boxtimes H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when ($g_1 g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$) or ($g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$).
  - In the *lexicographical product* $G \circ H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $g_1 g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$).

Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *strong product* $G \boxtimes H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when ($g_1 g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$) or ($g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$).
  - In the *lexicographical product* $G \circ H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $g_1 g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$).

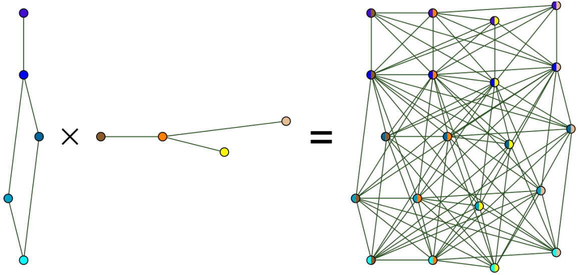
Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *strong product* $G \boxtimes H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when ($g_1 g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$) or ($g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$).
  - In the *lexicographical product* $G \circ H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $g_1 g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$).



Most common products of graphs

- The vertex set of the product of two graphs G and H is equal to $V(G) \times V(H)$ while their edge-sets are as follows:
  - In the *strong product* $G \boxtimes H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when ($g_1 g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$) or ($g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$).
  - In the *lexicographical product* $G \circ H$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent when $g_1 g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1 h_2 \in E(H)$).



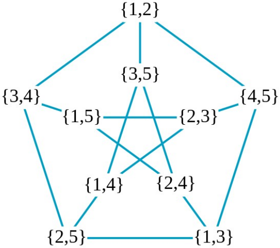
Kneser graphs

- [Erdős-Ko-Rado, 61] $\alpha(K(n,k)) = \binom{n-1}{k-1}$

- The Kneser graph $K(n,k)$ has as vertices all k-element subsets of the set $[n] = \{1,\ldots,n\}$ and an edge between two subsets if and only if they are disjoint.

Kneser graphs

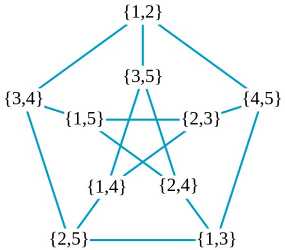
- The Kneser graph $K(n,k)$ has as vertices all k-element subsets of the set $[n] = \{1,\ldots,n\}$ and an edge between two subsets if and only if they are disjoint.



- [Erdős-Ko-Rado, 61] $\alpha(K(n,k)) = \binom{n-1}{k-1}$

Kneser graphs

- The Kneser graph $K(n, k)$ has as vertices all k-element subsets of the set $[n] = \{1, \ldots, n\}$ and an edge between two subsets if and only if they are disjoint.



- [Erdős-Ko-Rado, 61] $\alpha(K(n, k)) = \binom{n-1}{k-1}$

Known results

- [Geller and Stahl, 75] $\alpha(G \circ H) = \alpha(G) \times \alpha(H)$.

- [V-P and Vera, 06] Let $G = K(n_1, k_1)$ and $H = K(n_2, k_2)$. Then,

$$\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$$

Known results

- [Geller and Stahl, 75] $\alpha(G \circ H) = \alpha(G) \times \alpha(H)$.
- [V-P and Vera, 06] Let $G = K(n_1, k_1)$ and $H = K(n_2, k_2)$. Then,

$$\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$$

Strong product of Kneser graphs

- **Obs.** $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.
- The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

- [Lovász, 79] $\Theta(K(n,k)) = \binom{n-1}{k-1} \implies$

$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

- [Thm.] For any n and any $k \leq n/2$ we have $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.
- [Obs.]

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

Strong product of Kneser graphs

- Obs. $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.
- The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

- [Lovász, 79] $\Theta(K(n,k)) = \binom{n-1}{k-1} \implies$

$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

- [Thm.] For any n and any $k \leq n/2$ we have $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.
- [Obs.]

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

Strong product of Kneser graphs

- Obs. $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.
- The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

- [Lovász, 79] $\Theta(K(n,k)) = \binom{n-1}{k-1} \implies$

$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

- [Thm.] For any n and any $k \leq n/2$ we have $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.
- [Obs.]

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

Strong product of Kneser graphs

- Obs. $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.
- The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

- [Lovász, 79] $\Theta(K(n,k)) = \binom{n-1}{k-1} \implies$

$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

- [Thm.] For any n and any $k \leq n/2$ we have $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.
- [Obs.]

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

Strong product of Kneser graphs

- Obs. $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.
- The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

- [Lovász, 79] $\Theta(K(n,k)) = \binom{n-1}{k-1} \implies$

$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

- [Thm.] For any n and any $k \leq n/2$ we have $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.
- [Obs.]

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

Strong product of Kneser graphs

- Obs. $\alpha(K(n,k) \boxtimes K(n,k)) \geq (\alpha(K(n,k)))^2 = \binom{n-1}{k-1}^2$.
- The *Shannon capacity* $\Theta(G)$ of a graph G is defined as

$$\Theta(G) = \sup_{m \in \mathbb{N}} \sqrt[m]{\alpha(G^{m\boxtimes})}.$$

- [Lovász, 79] $\Theta(K(n,k)) = \binom{n-1}{k-1} \implies$

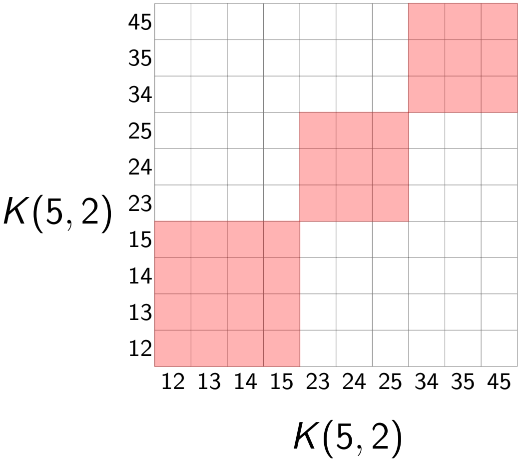
$$\sqrt{(\alpha(K(n,k) \boxtimes K(n,k))} \leq \binom{n-1}{k-1}.$$

- [Thm.] For any n and any $k \leq n/2$ we have $\alpha(K(n,k) \boxtimes K(n,k)) = \binom{n-1}{k-1}^2$.
- [Obs.]

$$\alpha((K(n,k)^{m\boxtimes}) = \binom{n-1}{k-1}^m.$$

Cartesian product of Kneser graphs $K(n,k)$

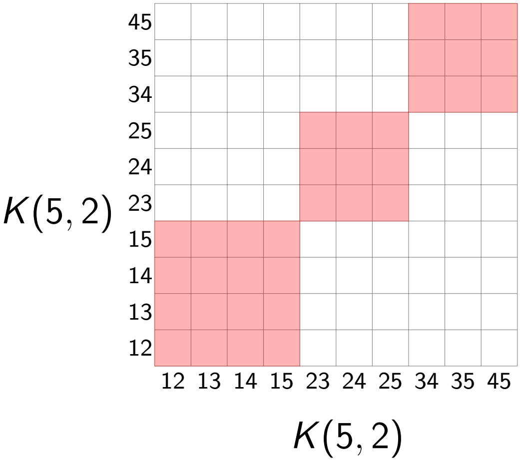
- Greedy diagonalisation procedure



- $\alpha(K(5,2)\square K(5,2)) = 34.$

Cartesian product of Kneser graphs $K(n,k)$

- Greedy diagonalisation procedure



- $\alpha(K(5,2)\square K(5,2)) = 34.$

Cartesian product of Kneser graphs $K(n, k)$

- Greedy diagonalisation procedure:
  $\alpha(K(n,k)\Box K(n,k)) \geq \binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \cdots + \binom{2k-1}{k-1}^2 + \binom{2k-1}{k-1}^2$
- Since $K(2k-1, k)$ is isomorphic to $\binom{2k-1}{k}$ isolated vertices then, $K(2k-1,k)\Box K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k-1}^2$ isolated vertices.
- By diagonalisation, $\alpha(K(6,2)\Box K(6,2)) \geq 59$, but $\alpha(K(6,2)\Box K(6,2)) \geq 60$
- Let $I(x) = \{A \in V(K(n,k)) : x \in A\}$. Then, $I(x)$ is an independence set in $K(n,k)$ with center in x.

Cartesian product of Kneser graphs $K(n,k)$

- Greedy diagonalisation procedure:
  $\alpha\big(K(n,k)\Box K(n,k)\big) \geq \binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \cdots + \binom{2k-1}{k-1}^2 + \binom{2k-1}{k-1}^2$
- Since $K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k}$ isolated vertices then, $K(2k-1,k)\Box K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k-1}^2$ isolated vertices.
- By diagonalisation, $\alpha(K(6,2)\Box K(6,2)) \geq 59$, but $\alpha(K(6,2)\Box K(6,2)) \geq 60$
- Let $I(x) = \{A \in V(K(n,k)) : x \in A\}$. Then, $I(x)$ is an independence set in $K(n,k)$ with center in x.

Cartesian product of Kneser graphs $K(n,k)$

- Greedy diagonalisation procedure:
  $\alpha\big(K(n,k)\square K(n,k)\big) \geq \binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \cdots + \binom{2k-1}{k-1}^2 + \binom{2k-1}{k-1}^2$
- Since $K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k}$ isolated vertices then, $K(2k-1,k)\square K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k-1}^2$ isolated vertices.
- By diagonalisation, $\alpha(K(6,2)\square K(6,2)) \geq 59$, but $\alpha(K(6,2)\square K(6,2)) \geq 60$
- Let $I(x) = \{A \in V(K(n,k)) : x \in A\}$. Then, $I(x)$ is an independence set in $K(n,k)$ with center in x.

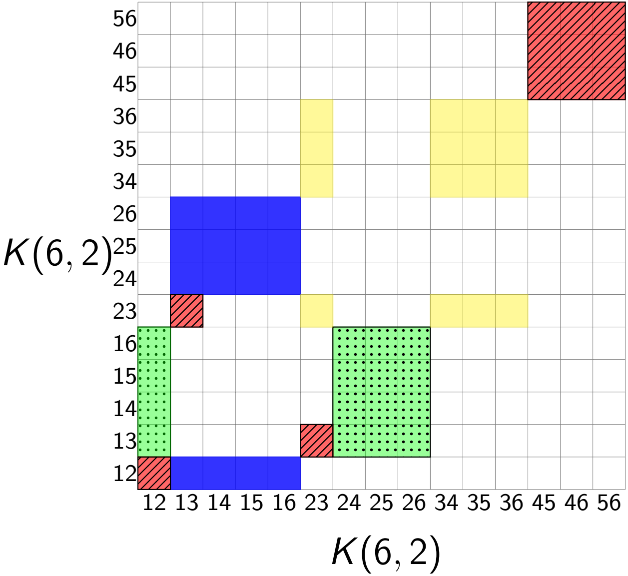
Cartesian product of Kneser graphs $K(n,k)$

- Greedy diagonalisation procedure:
  $\alpha\big(K(n,k)\square K(n,k)\big) \geq \binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \cdots + \binom{2k-1}{k-1}^2 + \binom{2k-1}{k-1}^2$
- Since $K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k}$ isolated vertices then, $K(2k-1,k)\square K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k-1}^2$ isolated vertices.
- By diagonalisation, $\alpha(K(6,2)\square K(6,2)) \geq 59$, but $\alpha(K(6,2)\square K(6,2)) \geq 60$
- Let $I(x) = \{A \in V(K(n,k)) : x \in A\}$. Then, $I(x)$ is an independence set in $K(n,k)$ with center in x.

Cartesian product of Kneser graphs $K(n,k)$

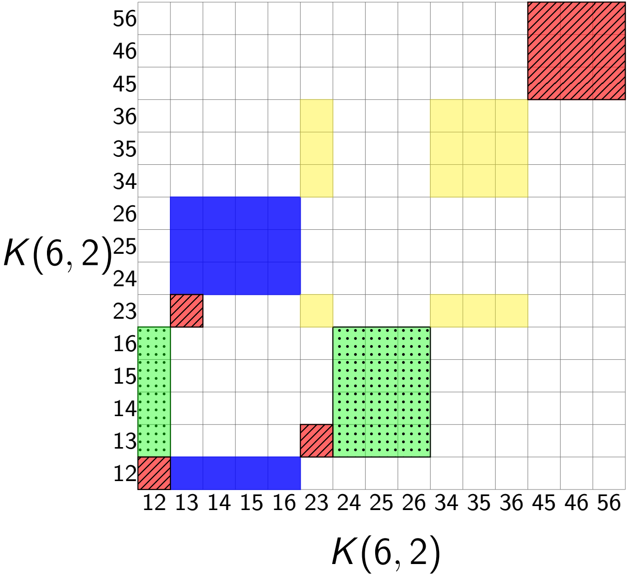
- Greedy diagonalisation procedure:
  $\alpha\big(K(n,k)\square K(n,k)\big) \geq \binom{n-1}{k-1}^2 + \binom{n-2}{k-1}^2 + \cdots + \binom{2k-1}{k-1}^2 + \binom{2k-1}{k-1}^2$
- Since $K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k}$ isolated vertices then, $K(2k-1,k)\square K(2k-1,k)$ is isomorphic to $\binom{2k-1}{k-1}^2$ isolated vertices.
- By diagonalisation, $\alpha(K(6,2)\square K(6,2)) \geq 59$, but $\alpha(K(6,2)\square K(6,2)) \geq 60$
- Let $I(x) = \{A \in V(K(n,k)) : x \in A\}$. Then, $I(x)$ is an independence set in $K(n,k)$ with center in x.

Cartesian product of Kneser graphs $K(n,2)$



$(I(1_2) \times I(2_3)) \cup (I(2_3) \times I(1_2)) \cup (I(3_1) \times I(3_1)) \cup \{(12,12),(13,23),(23,13)\}$

Cartesian product of Kneser graphs $K(n, 2)$



- $(I(1_2) \times I(2_3)) \cup (I(2_3) \times I(1_2)) \cup (I(3_1) \times I(3_1)) \cup \{(12, 12), (13, 23), (23, 13)\}$

Cartesian product of Kneser graphs $K(n, 2)$

- After dealing with the subgraph of $K(n,2)\square K(n,2)$ induced by $(I(1)\cup I(2)\cup I(3))\times(I(1)\cup I(2)\cup I(3))$, reduces to the independence number of $K(n-3,2)\square K(n-3,2)$ (as soon as $n\geq 6$). By using a simple induction with respect to three different bases, $\alpha(K(3,2)^{2\square})=9, \alpha(K(4,2)^{2\square})=18$, and $\alpha(K(5,2)^{2\square})=34$, we obtain the following result.
- Proposition. Let $n\geq 6$, and $n\equiv t \pmod 3$. Then

$$\alpha(K(n,2)\square K(n,2))\geq 3(n-2)^2+3(n-5)^2+\cdots+3(t+4)^2+(n-t-3)+A_t,$$

where

$$A_t=\begin{cases} 9 & ;\ t=0 \\ 18 & ;\ t=1 \\ 34 & ;\ t=2. \end{cases}$$

- This bound is bigger by $(n-t-1)/3$ than the greedy diagonalization bound.

Cartesian product of Kneser graphs $K(n,2)$

- After dealing with the subgraph of $K(n,2)\square K(n,2)$ induced by $(I(1)\cup I(2)\cup I(3))\times(I(1)\cup I(2)\cup I(3))$, reduces to the independence number of $K(n-3,2)\square K(n-3,2)$ (as soon as $n\geq 6$). By using a simple induction with respect to three different bases, $\alpha(K(3,2)^{2\square})=9, \alpha(K(4,2)^{2\square})=18$, and $\alpha(K(5,2)^{2\square})=34$, we obtain the following result.
- Proposition. Let $n\geq 6$, and $n\equiv t \pmod 3$. Then

$$\alpha(K(n,2)\square K(n,2))\geq 3(n-2)^2+3(n-5)^2+\cdots+3(t+4)^2+(n-t-3)+A_t,$$

where

$$A_t=\begin{cases} 9 & ;\ t=0\\ 18 & ;\ t=1\\ 34 & ;\ t=2.\end{cases}$$

- This bound is bigger by $(n-t-1)/3$ than the greedy diagonalization bound.

Cartesian product of Kneser graphs $K(n,2)$

- After dealing with the subgraph of $K(n,2)\square K(n,2)$ induced by $(I(1)\cup I(2)\cup I(3))\times(I(1)\cup I(2)\cup I(3))$, reduces to the independence number of $K(n-3,2)\square K(n-3,2)$ (as soon as $n\geq 6$). By using a simple induction with respect to three different bases, $\alpha(K(3,2)^{2\square})=9, \alpha(K(4,2)^{2\square})=18$, and $\alpha(K(5,2)^{2\square})=34$, we obtain the following result.
- Proposition. Let $n\geq 6$, and $n\equiv t \pmod 3$. Then

$$\alpha(K(n,2)\square K(n,2))\geq 3(n-2)^2+3(n-5)^2+\cdots+3(t+4)^2+(n-t-3)+A_t,$$

where

$$A_t=\begin{cases} 9 & ;\ t=0\\ 18 & ;\ t=1\\ 34 & ;\ t=2.\end{cases}$$

- This bound is bigger by $(n-t-1)/3$ than the greedy diagonalization bound.

Cartesian product of Kneser graphs $K(n,2)$

- **Corollary.** If $n \geq 3$, then

$$\alpha(K(n,2)\Box K(n,2)) \geq \begin{cases} \frac{2n^3-3n^2+3n+18}{6} & ; \quad n \equiv 0 \pmod 3 \\ \frac{2n^3-3n^2+3n+16}{6} & ; \quad n \equiv 1 \pmod 3 \\ \frac{2n^3-3n^2+3n+14}{6} & ; \quad n \equiv 2 \pmod 3. \end{cases}$$

- **Problem:** A better lower bound for $\alpha(K(n,2)\Box K(n,2))$?

Cartesian product of Kneser graphs $K(n,2)$

- **Corollary.** If $n \geq 3$, then

$$\alpha(K(n,2)\Box K(n,2)) \geq \begin{cases} \frac{2n^3-3n^2+3n+18}{6} & ; \quad n \equiv 0 \pmod{3} \\ \frac{2n^3-3n^2+3n+16}{6} & ; \quad n \equiv 1 \pmod{3} \\ \frac{2n^3-3n^2+3n+14}{6} & ; \quad n \equiv 2 \pmod{3}. \end{cases}$$

- **Problem:** A better lower bound for $\alpha(K(n,2)\Box K(n,2))$?

Upper bounds for $\alpha(K(n,2)^{\square 2})$

- Let H be a graph and let a A_i, $1 \leq i \leq m$, a partition of $V(H)$, where each subset A_i induces a clique in H, with $l_i = |A_i|$.
- [Klavžar, 05] $\alpha(G \square H) \leq \sum_{i=1}^{m} \alpha_{\ell_i}(G)$, where $\alpha_{\ell_i}(G)$ denotes the ℓ_i-independence number of a graph G.
- [Baranyai, 75] $K(n,k)$ can be partitioned into $\theta_{n,k} = \left\lceil \binom{n}{k} / \lfloor n/k \rfloor \right\rceil$ cliques of size $\omega_{n,k} = \lfloor n/k \rfloor$.
- Theorem. $\alpha(K(n,k) \square K(n,k)) \leq \left\lceil \frac{\binom{n}{k}}{\lfloor \frac{n}{k} \rfloor} \right\rceil \alpha_{\omega_{n,k}}(K(n,k))$.

Upper bounds for $\alpha(K(n,2)^{\square 2})$

- Let H be a graph and let a A_i, $1 \leq i \leq m$, a partition of $V(H)$, where each subset A_i induces a clique in H, with $l_i = |A_i|$.
- [Klavžar, 05] $\alpha(G \square H) \leq \sum_{i=1}^{m} \alpha_{\ell_i}(G)$, where $\alpha_{\ell_i}(G)$ denotes the ℓ_i-independence number of a graph G.
- [Baranyai, 75] $K(n,k)$ can be partitioned into $\theta_{n,k} = \left\lceil \binom{n}{k} / \lfloor n/k \rfloor \right\rceil$ cliques of size $\omega_{n,k} = \lfloor n/k \rfloor$.
- Theorem. $\alpha(K(n,k) \square K(n,k)) \leq \left\lceil \frac{\binom{n}{k}}{\lfloor \frac{n}{k} \rfloor} \right\rceil \alpha_{\omega_{n,k}}(K(n,k))$.

Upper bounds for $\alpha(K(n,2)^{\square 2})$

- Let H be a graph and let a A_i, $1 \leq i \leq m$, a partition of $V(H)$, where each subset A_i induces a clique in H, with $l_i = |A_i|$.
- [Klavžar, 05] $\alpha(G \square H) \leq \sum_{i=1}^{m} \alpha_{\ell_i}(G)$, where $\alpha_{\ell_i}(G)$ denotes the ℓ_i-independence number of a graph G.
- [Baranyai, 75] $K(n,k)$ can be partitioned into $\theta_{n,k} = \left\lceil \binom{n}{k} / \lfloor n/k \rfloor \right\rceil$ cliques of size $\omega_{n,k} = \lfloor n/k \rfloor$.
- Theorem. $\alpha(K(n,k) \square K(n,k)) \leq \left\lceil \frac{\binom{n}{k}}{\lfloor \frac{n}{k} \rfloor} \right\rceil \alpha_{\omega_{n,k}}(K(n,k))$.

Upper bounds for $\alpha(K(n,2)^{\square 2})$

- Let H be a graph and let a A_i, $1 \leq i \leq m$, a partition of $V(H)$, where each subset A_i induces a clique in H, with $l_i = |A_i|$.
- [Klavžar, 05] $\alpha(G \square H) \leq \sum_{i=1}^{m} \alpha_{\ell_i}(G)$, where $\alpha_{\ell_i}(G)$ denotes the ℓ_i-independence number of a graph G.
- [Baranyai, 75] $K(n,k)$ can be partitioned into $\theta_{n,k} = \left\lceil \binom{n}{k} / \lfloor n/k \rfloor \right\rceil$ cliques of size $\omega_{n,k} = \lfloor n/k \rfloor$.
- Theorem. $\alpha(K(n,k) \square K(n,k)) \leq \left\lceil \frac{\binom{n}{k}}{\lfloor \frac{n}{k} \rfloor} \right\rceil \alpha_{\omega_{n,k}}(K(n,k))$.

l-independence number of $K(n,2)$

- **Proposition.** For $n \geq 5$,

$$\alpha_\ell(K(n,2)) = \begin{cases} (n-1)+\cdots+(n-\ell) & ; \quad \ell < n-2 \\ \binom{n}{2} & ; \quad \ell \geq n-2\,. \end{cases}$$

- **Theorem.** If $n \geq 5$, then

$$\left.\begin{array}{lll} \frac{2n^3-3n^2+3n+18}{6} & ; & n \equiv 0 \pmod 3 \\ \frac{2n^3-3n^2+3n+16}{6} & ; & n \equiv 1 \pmod 3 \\ \frac{2n^3-3n^2+3n+14}{6} & ; & n \equiv 2 \pmod 3). \end{array}\right\} \leq \alpha(K(n,2)\Box K(n,2)) \leq$$

$$\begin{cases} \frac{n(n-1)(3n-2)}{8} & ; \quad n \text{ even} \\ \frac{n(n-1)(3n-1)}{8} & ; \quad n \text{ odd.} \end{cases}$$

l-independence number of $K(n,2)$

- **Proposition.** For $n \geq 5$,

$$\alpha_\ell(K(n,2)) = \begin{cases} (n-1)+\cdots+(n-\ell) & ; \quad \ell < n-2 \\ \binom{n}{2} & ; \quad \ell \geq n-2\,. \end{cases}$$

- **Theorem.** If $n \geq 5$, then

$$\left.\begin{array}{ll} \frac{2n^3-3n^2+3n+18}{6} & ; \quad n \equiv 0 \pmod 3 \\ \frac{2n^3-3n^2+3n+16}{6} & ; \quad n \equiv 1 \pmod 3 \\ \frac{2n^3-3n^2+3n+14}{6} & ; \quad n \equiv 2 \pmod 3. \end{array}\right\} \leq \alpha(K(n,2)\Box K(n,2)) \leq$$

$$\begin{cases} \frac{n(n-1)(3n-2)}{8} & ; \quad n \text{ even} \\ \frac{n(n-1)(3n-1)}{8} & ; \quad n \text{ odd.} \end{cases}$$

2-independence number of $K(2k+1,k)$

- [Godsil, 04] $\alpha_2(K(2k+1,k)) \leq \binom{2k}{k} + \binom{2k}{k-2}$.
- $\alpha_2(K(5,2)) \leq 7$, $\alpha_2(K(7,3)) \leq 26$ and $\alpha_2(K(9,4)) \leq 98$.
- Proposition. For $k \geq 2$,

$$\alpha_2(K(2k+1,k)) \geq \begin{cases} \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 & \text{, if } k \text{ even} \\ \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2 & \text{, if } k \text{ odd} \end{cases}$$

- $\alpha_2(K(5,2)) \geq 7$, $\alpha_2(K(7,3)) \geq 26$ and $\alpha_2(K(9,4)) \geq 96$.

2-independence number of $K(2k+1,k)$

- [Godsil, 04] $\alpha_2(K(2k+1,k)) \leq \binom{2k}{k} + \binom{2k}{k-2}$.
- $\alpha_2(K(5,2)) \leq 7$, $\alpha_2(K(7,3)) \leq 26$ and $\alpha_2(K(9,4)) \leq 98$.
- Proposition. For $k \geq 2$,

$$\alpha_2(K(2k+1,k)) \geq \begin{cases} \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 & \text{, if } k \text{ even} \\ \\ \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2 & \text{, if } k \text{ odd} \end{cases}$$

- $\alpha_2(K(5,2)) \geq 7$, $\alpha_2(K(7,3)) \geq 26$ and $\alpha_2(K(9,4)) \geq 96$.

2-independence number of $K(2k+1,k)$

- [Godsil, 04] $\alpha_2(K(2k+1,k)) \leq \binom{2k}{k} + \binom{2k}{k-2}$.
- $\alpha_2(K(5,2)) \leq 7$, $\alpha_2(K(7,3)) \leq 26$ and $\alpha_2(K(9,4)) \leq 98$.

- Proposition. For $k \geq 2$,

$$\alpha_2(K(2k+1,k)) \geq \begin{cases} \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 & \text{, if } k \text{ even} \\ \\ \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2 & \text{, if } k \text{ odd} \end{cases}$$

- $\alpha_2(K(5,2)) \geq 7$, $\alpha_2(K(7,3)) \geq 26$ and $\alpha_2(K(9,4)) \geq 96$.

2-independence number of $K(2k+1,k)$

- [Godsil, 04] $\alpha_2(K(2k+1,k)) \leq \binom{2k}{k} + \binom{2k}{k-2}$.
- $\alpha_2(K(5,2)) \leq 7$, $\alpha_2(K(7,3)) \leq 26$ and $\alpha_2(K(9,4)) \leq 98$.

- Proposition. For $k \geq 2$,

$$\alpha_2(K(2k+1,k)) \geq \begin{cases} \binom{2k+1}{k} - \frac{(k+1)}{(k+2)}\binom{k}{k/2}^2 & \text{, if } k \text{ even} \\ \\ \binom{2k+1}{k} - \binom{k}{\lfloor k/2 \rfloor}^2 & \text{, if } k \text{ odd} \end{cases}$$

- $\alpha_2(K(5,2)) \geq 7$, $\alpha_2(K(7,3)) \geq 26$ and $\alpha_2(K(9,4)) \geq 96$.

2-independence number of $K(2k+1,k)$

- Let $k \geq 2$. Let $x_0 = [k]$. For $0 \leq i \leq k$, let X_i be the set of vertices of $K(2k+1,k)$ defined as: $X_0 = \{x_0\}$ and $X_i = \{y : y \in V(K(2k+1,k)), |x_0 \cap y| = i-1\}$. Note that $\bigcup_{i=0}^{k} X_i$ forms a partition of the vertex set of $K(2k+1,k)$.
- Lemma. Let G_i be the subgraph of $K(2k+1,k)$ induced by the vertex set X_i, for $0 \leq i \leq k$. Then,
  1. For $0 \leq i \leq k$, with $i \neq \lfloor \frac{k}{2} \rfloor + 1$, the subgraph G_i induces an independent set.

2-independence number of $K(2k+1,k)$

- Let $k \geq 2$. Let $x_0 = [k]$. For $0 \leq i \leq k$, let X_i be the set of vertices of $K(2k+1,k)$ defined as: $X_0 = \{x_0\}$ and $X_i = \{y : y \in V(K(2k+1,k)), |x_0 \cap y| = i-1\}$. Note that $\bigcup_{i=0}^{k} X_i$ forms a partition of the vertex set of $K(2k+1,k)$.
- Lemma. Let G_i be the subgraph of $K(2k+1,k)$ induced by the vertex set X_i, for $0 \leq i \leq k$. Then,

1. For $0 \leq i \leq k$, with $i \neq \left\lfloor \frac{k}{2} \right\rfloor + 1$, the subgraph G_i induces an independent set.
2. Let $G_A = \bigcup_{1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor} G_i$ and let $G_B = \bigcup_{\left\lfloor \frac{k}{2} \right\rfloor + 2 \leq i \leq k} G_i$. Then, the subgraph G_A (resp. G_B) induces an independent set.
3. Let $i = \left\lfloor \frac{k}{2} \right\rfloor + 1$. The subgraph G_i is isomorphic to $\frac{1}{2}\binom{k}{k/2}$ disjoint copies of the direct product graph $K_2 \times K(k+1, k/2)$ if k is even, and G_i is isomorphic to $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$ disjoint copies of the direct product graph $K(k, \lfloor k/2 \rfloor) \times K_2$ if k is odd.

2-independence number of $K(2k+1,k)$

- Let $k \geq 2$. Let $x_0 = [k]$. For $0 \leq i \leq k$, let X_i be the set of vertices of $K(2k+1,k)$ defined as: $X_0 = \{x_0\}$ and $X_i = \{y : y \in V(K(2k+1,k)), |x_0 \cap y| = i-1\}$. Note that $\bigcup_{i=0}^{k} X_i$ forms a partition of the vertex set of $K(2k+1,k)$.
- Lemma. Let G_i be the subgraph of $K(2k+1,k)$ induced by the vertex set X_i, for $0 \leq i \leq k$. Then,
    1. For $0 \leq i \leq k$, with $i \neq \left\lfloor \frac{k}{2} \right\rfloor + 1$, the subgraph G_i induces an independent set.
    2. Let $G_A = \bigcup_{1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor} G_i$ and let $G_B = \bigcup_{\left\lfloor \frac{k}{2} \right\rfloor + 2 \leq i \leq k} G_i$. Then, the subgraph G_A (resp. G_B) induces an independent set.
    3. Let $i = \left\lfloor \frac{k}{2} \right\rfloor + 1$. The subgraph G_i is isomorphic to $\frac{1}{2}\binom{k}{k/2}$ disjoint copies of the direct product graph $K_2 \times K(k+1, k/2)$ if k is even, and G_i is isomorphic to $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$ disjoint copies of the direct product graph $K(k, \lfloor k/2 \rfloor) \times K_2$ if k is odd.

2-independence number of $K(2k+1,k)$

- Let $k \geq 2$. Let $x_0 = [k]$. For $0 \leq i \leq k$, let X_i be the set of vertices of $K(2k+1,k)$ defined as: $X_0 = \{x_0\}$ and $X_i = \{y : y \in V(K(2k+1,k)), |x_0 \cap y| = i-1\}$. Note that $\bigcup_{i=0}^{k} X_i$ forms a partition of the vertex set of $K(2k+1,k)$.
- Lemma. Let G_i be the subgraph of $K(2k+1,k)$ induced by the vertex set X_i, for $0 \leq i \leq k$. Then,
  1. For $0 \leq i \leq k$, with $i \neq \left\lfloor \frac{k}{2} \right\rfloor + 1$, the subgraph G_i induces an independent set.
  2. Let $G_A = \bigcup_{1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor} G_i$ and let $G_B = \bigcup_{\left\lfloor \frac{k}{2} \right\rfloor + 2 \leq i \leq k} G_i$. Then, the subgraph G_A (resp. G_B) induces an independent set.
  3. Let $i = \left\lfloor \frac{k}{2} \right\rfloor + 1$. The subgraph G_i is isomorphic to $\frac{1}{2}\binom{k}{k/2}$ disjoint copies of the direct product graph $K_2 \times K(k+1, k/2)$ if k is even, and G_i is isomorphic to $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$ disjoint copies of the direct product graph $K(k, \lfloor k/2 \rfloor) \times K_2$ if k is odd.

2-independence number of $K(2k+1, k)$

- Let $k \geq 2$. Let $x_0 = [k]$. For $0 \leq i \leq k$, let X_i be the set of vertices of $K(2k+1, k)$ defined as: $X_0 = \{x_0\}$ and $X_i = \{y : y \in V(K(2k+1, k)), |x_0 \cap y| = i - 1\}$. Note that $\bigcup_{i=0}^{k} X_i$ forms a partition of the vertex set of $K(2k+1, k)$.
- Lemma. Let G_i be the subgraph of $K(2k+1, k)$ induced by the vertex set X_i, for $0 \leq i \leq k$. Then,
  1. For $0 \leq i \leq k$, with $i \neq \left\lfloor \frac{k}{2} \right\rfloor + 1$, the subgraph G_i induces an independent set.
  2. Let $G_A = \bigcup_{1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor} G_i$ and let $G_B = \bigcup_{\left\lfloor \frac{k}{2} \right\rfloor + 2 \leq i \leq k} G_i$. Then, the subgraph G_A (resp. G_B) induces an independent set.
  3. Let $i = \left\lfloor \frac{k}{2} \right\rfloor + 1$. The subgraph G_i is isomorphic to $\frac{1}{2}\binom{k}{k/2}$ disjoint copies of the direct product graph $K_2 \times K(k+1, k/2)$ if k is even, and G_i is isomorphic to $\frac{1}{2}\binom{k+1}{\lceil k/2 \rceil}$ disjoint copies of the direct product graph $K(k, \lfloor k/2 \rfloor) \times K_2$ if k is odd.

2-independence number of $K(2k+1,k)$

- Let G_A and G_B the subgraphs of $K(2k+1,k)$ as defined before. Let G_C be the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor +1}$. Let k even, then
- Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, i.e., $Y_1 = G_C \cap I(1)$.
- Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$.
- x_0 is neither adjacent to any vertex in G_B nor in Y_1.
- Let $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2 + 1 \leq j \leq k-1$, and we must have that $k/2 + j \leq k$, which is impossible.

2-independence number of $K(2k+1,k)$

- Let G_A and G_B the subgraphs of $K(2k+1,k)$ as defined before. Let G_C be the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor +1}$. Let k even, then
- Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, i.e., $Y_1 = G_C \cap I(1)$.
- Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$.
- x_0 is neither adjacent to any vertex in G_B nor in Y_1.
- Let $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2 + 1 \leq j \leq k-1$, and we must have that $k/2 + j \leq k$, which is impossible.

2-independence number of $K(2k+1,k)$

- Let G_A and G_B the subgraphs of $K(2k+1,k)$ as defined before. Let G_C be the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor +1}$. Let k even, then
- Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, i.e., $Y_1 = G_C \cap I(1)$.
- Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$.
- x_0 is neither adjacent to any vertex in G_B nor in Y_1.
- Let $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2+1 \leq j \leq k-1$, and we must have that $k/2 + j \leq k$, which is impossible.

2-independence number of $K(2k+1, k)$

- Let G_A and G_B the subgraphs of $K(2k+1, k)$ as defined before. Let G_C be the subgraph of $K(2k+1, k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor + 1}$. Let k even, then
- Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, i.e., $Y_1 = G_C \cap I(1)$.
- Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$.
- x_0 is neither adjacent to any vertex in G_B nor in Y_1.
- Let $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2 + 1 \leq j \leq k - 1$, and we must have that $k/2 + j \leq k$, which is impossible.

2-independence number of $K(2k+1,k)$

- Let G_A and G_B the subgraphs of $K(2k+1,k)$ as defined before. Let G_C be the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor+1}$. Let k even, then
- Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, i.e., $Y_1 = G_C \cap I(1)$.
- Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$.
- x_0 is neither adjacent to any vertex in G_B nor in Y_1.
- Let $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2+1 \leq j \leq k-1$, and we must have that $k/2 + j \leq k$, which is impossible.

2-independence number of $K(2k+1,k)$

- Let G_A and G_B the subgraphs of $K(2k+1,k)$ as defined before. Let G_C be the subgraph of $K(2k+1,k)$ induced by the vertices in the set $X_{\lfloor k/2 \rfloor +1}$. Let k even, then
- Let $Y_1 = \{y : y \in G_C \text{ and } 1 \in y\}$, i.e., $Y_1 = G_C \cap I(1)$.
- Set $W_1 = \{x_0\} \cup \{y : y \in G_B\} \cup Y_1$ and $W_2 = \{y : y \in G_A\}$.
- x_0 is neither adjacent to any vertex in G_B nor in Y_1.
- Let $a \in Y_1$ and $b \in G_B$ such that a is adjacent to b. By construction, $|a \cap x_0| = k/2$ and $|b \cap x_0| = j$, with $k/2 + 1 \leq j \leq k-1$, and we must have that $k/2 + j \leq k$, which is impossible.

Thank you !