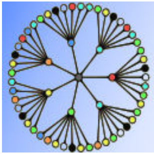

SOME INSIGHT ON GRAPHS WITH PERFECT CLOSED NEIGHBOURHOOD MATRICES

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CONICET

joint work with M. Escalante and E. Hinrichsen

11 y 12 diciembre 2018

IX Seminario de la Red Latinoamericana de Optimización Discreta y Grafos

Bahía Blanca- Argentina

Characterize graphs with perfect closed neighborhood matrices

Characterize graphs with perfect closed neighborhood matrices

Why?

MOTIVATION: THE $\{k\}$-PACKING FUNCTION PROBLEM (FIXED k)

[HINRICHSEN ET AL.. 2014]

$f : V \longrightarrow \{0, 1, \cdots, k\}$ is $\{k\}$-packing function of a graph G if

$$\forall v \in V, \sum_{w \in N[v]} f(w) \leq k.$$

$$L_{\{k\}}(G) = \max \left\{ \sum_{v \in V} f(v) : \{k\}\text{-packing function } f \text{ of } G \right\}$$

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generalizes the notion of k-limited packing [Gallant et al., 2010]:

- $f : V \longrightarrow \{0,1\}$ is *k-limited packing* of a graph G if

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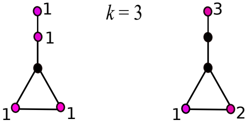

[Hinrichsen et al.. 2014]

$4 \quad < \quad L_{\{3\}}(G) \; = \; 6$

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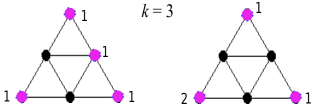

[Hinrichsen et al.. 2014]

$$4 \quad = \quad L_{\{3\}}(G)$$

COMPLEXITY RESULTS FOR THE $\{k\}$-PACKING FUNCTION PROBLEM

1. Linear time solvable for **strongly chordal graphs**. [Hinrichsen, L.- 2014]
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3. LinEMSOL $\rightarrow$ linear for graphs with cwd bounded by a constant. [Milanič, Hinrichsen, L.- 2014]
4. NP-complete for bipartite graphs of diameter 10. [Private communication with Safe M., 2016]

MOTIVATION: THE $\{k\}$-PACKING FUNCTION PROBLEM (FIXED k)

It is clear that $L_{\{k\}}(G) \geq k\, L_{\{1\}}(G)$, for each k and every G.

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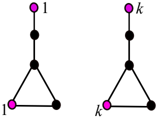
Given a $\{1\}$-packing function $f : V(G) \longrightarrow \{0,1\}$ of G, i.e.

$$\forall v \in V(G),\ f(N[v]) \leq 1,$$

let us define the following $\{k\}$-packing function $g : V(G) \longrightarrow \{0,1,\cdots,k\}$:

$$g(v) = \begin{cases} k & \text{if } f(v) = 1 \\ 0 & \text{if } f(v) = 0 \end{cases}$$



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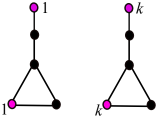
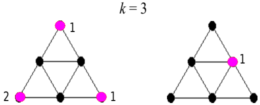




$$L_{\{3\}}(G) > 3\, L_{\{1\}}(G)$$

THEOREM

If G is such that $N(G)$ is a perfect matrix, then $L_{\{k\}}(G) = k\, L_1(G)$.

A $0,1$-matrix M having n (non zero) columns is *perfect* if all extreme points of
$P(M) = \{x \in [0,1]^n : M \cdot x \leq \mathbf{1}\}$
are $0,1$-vectors.

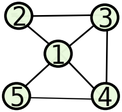
$$M \text{ perfect } \Rightarrow \max\ \{\mathbf{1} \cdot x : M \cdot x \leq \mathbf{1},\ x \in \{0,1\}^n\} = \max\ \{\mathbf{1} \cdot x :\ x \in P(M)\}.$$

Closed neighborhood matrices

$G = (V, E)$ simple graph

$N(G) = (a_{ij})_{|V|\times|V|}$ *closed neighborhood matrix* of G:

$$a_{ij} = \begin{cases} 1 & \text{if } i \in N[j] \\ 0 & \text{otherwise} \end{cases}$$



$$N(G) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

AN INTEGER LINEAR PROGRAM FOR THE $\{k\}$-PACKING FUNCTION PROBLEM

Recall

$$L_{\{k\}}(G) = \max\{f(V) : \ f : V \to \{0,1,\ldots,k\} \ \text{s.t.}\, f(N[v]) \leq k, \ \forall v \in V\}.$$

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- $G = (V,E)$, $|V| = n$, variables $x_i = f(i)$ for $i = 1,\ldots,n$
- $\mathbf{1}$: vector of all 1's

$$L_{\{k\}}(G) = max \quad \mathbf{1} \cdot x$$

subject to

$$N(G) \cdot x \leq k \cdot \mathbf{1}$$

$$x \in \{0,1,\ldots,k\}^n$$

An integer linear program for the $\{k\}$-packing function problem

Recall

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$$\sout{x \in \{0,1,\ldots,k\}^n} \quad x \in [0,k]^n$$

A sufficient condition for optimality

$$(1) \quad \begin{array}{rl} & max \;\; \mathbf{1} \cdot y \\ s.t. & N(G) \cdot y \leq \mathbf{1} \\ & y \in [0,1]^n \end{array} \qquad\qquad (2) \quad \begin{array}{rl} & max \;\; \mathbf{1} \cdot x \\ s.t. & N(G) \cdot x \leq k\,\mathbf{1} \\ & x \in [0,k]^n \end{array}$$

- y feasible for (1) $\longleftrightarrow$ $k\,N(G) \cdot y \leq k\,\mathbf{1}$ $\longleftrightarrow$ $x = k\,y$ feasible for (2).

$$(1)\quad \begin{array}{rl} L_1^R(G) = max & \mathbf{1}\cdot y \\ s.t. & N(G)\cdot y \leq \mathbf{1} \\ & y \in [0,1]^n \end{array} \qquad\qquad (2)\quad \begin{array}{rl} & max\ \ \mathbf{1}\cdot x \\ s.t. & N(G)\cdot x \leq k\,\mathbf{1} \\ & x \in [0,k]^n \end{array}$$

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- $L_{\{k\}}(G) \leq k\,max\{\mathbf{1}\cdot y : N(G)\cdot y \leq \mathbf{1},\ y \in [0,1]^n\} = k\,L_1^R(G)$.

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A SUFFICIENT CONDITION FOR OPTIMALITY

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THEOREM

If $N(G)$ is perfect then $L_{\{k\}}(G) = k\, L_1(G)$.

▷ The sufficient condition is not necessary:

$L_{\{k\}}(C_4) = k$ when k is not a multiple of 3, and $L_1(C_4) = 1$ but $N(C_4)$ is not perfect.

$$N(C_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

THEOREM [CHVÁTAL, 1975]

A $0,1$-matrix M with no zero columns is perfect if and only if it is the clique-node matrix of the graph $G_Q(M)$ (the *clique graph* of M) and $G_Q(M)$ is a perfect graph.

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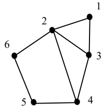
This is the clique-node matrix of



$G_Q(M)$, *perfect* graph, since it has neither induced odd cycles nor odd anticycles [Chudnowsky, 2003].

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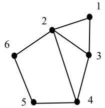

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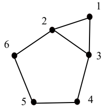
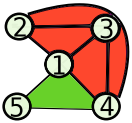


This is the clique-node matrix of $G_Q(M)$, *non perfect* graph, since it has an induced odd cycle.

$G = (V, E)$ simple graph

The *clique-node* matrix of G, $\mathscr{C}(G)$: columns indexed by V and rows are the incidence vectors of maximal cliques in G.



$$\mathscr{C}(G) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

CLIQUE-NODE MATRICES

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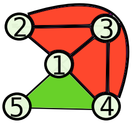
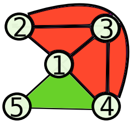


Why not $\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$?

$G = (V, E)$ simple graph

M is an *extended clique-node* matrix of G: $|V| \times |V|$, the clique-node is a row submatrix, the remaining rows correspond to cliques in G.



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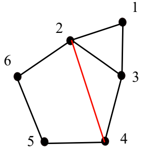
REFORMULATING CHVÁTAL'S THEOREM

A **square** $0,1$-matrix M with no zero columns is perfect if and only if it is an extended clique-node matrix of the graph $G_Q(M)$ (the *clique graph* of M) and $G_Q(M)$ is a perfect graph.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$



an extended clique-node matrix of the perfect graph $G_Q(M)$.

THEOREM (REFORMULATION IN TERMS OF EXTENDED CLIQUE-NODE MATRICES)

The following statements are equivalent:

1. *M is an extended clique-node matrix,*
2. *if $J - I$ is a $p \times p$ submatrix of M, where $p \geq 3$,*

$$J - I = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$$

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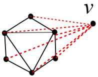
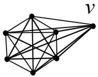
then M contains a row i such that $m_{ij} = 1$ for every column j of $J - I$.

Characterize graphs in

$$\mathscr{F} = \{G : N(G) \text{ is perfect}\}$$

Clearly, if a graph has a universal node then it belongs to $\mathscr{F}$:

G 

$\longrightarrow$ $G_Q(N(G)) = K_n$ 

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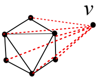
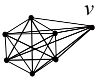
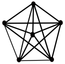
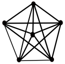
G 

$\longrightarrow$ $G_Q(N(G)) = K_n$ 

Examples are:

1. Complete graphs are in $\mathscr{F}$:

K_n  $\longrightarrow$ $G_Q(N(K_n)) = K_n$ 

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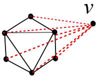
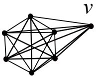
G 

$\longrightarrow$ $G_Q(N(G)) = K_n$ 

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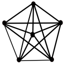
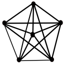
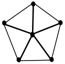
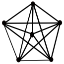
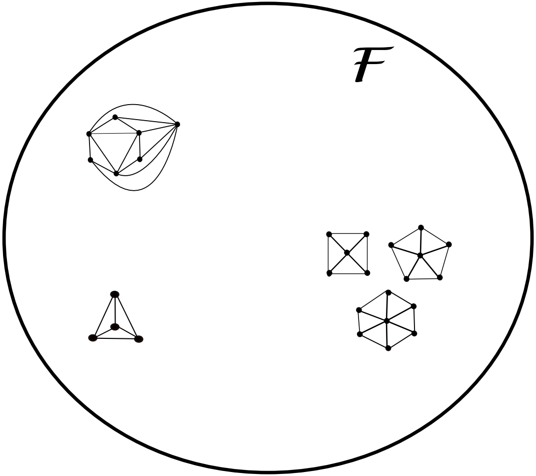
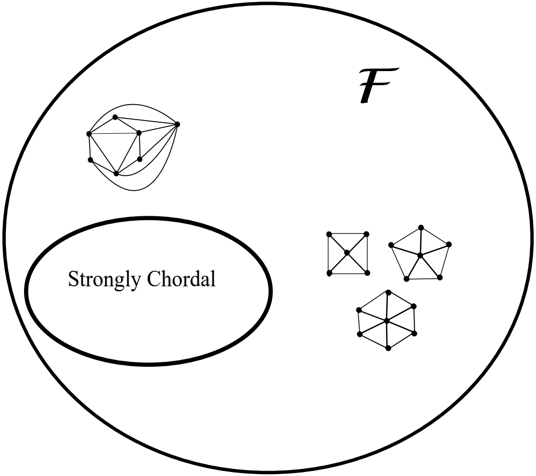
K_n  $\longrightarrow$ $G_Q(N(K_n)) = K_n$ 

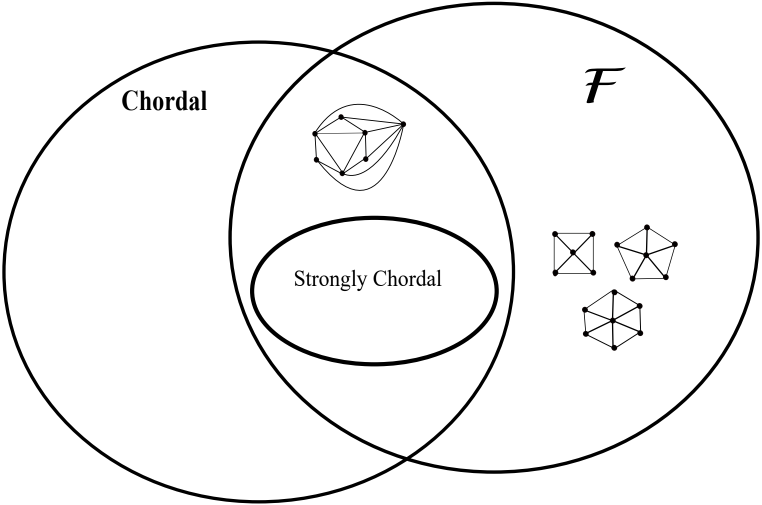
2. Wheels are in $\mathscr{F}$:

W_n  $\longrightarrow$ $G_Q(N(W_n)) = K_n$ 

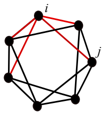

$\mathcal{F}$



- ▷ G is strongly chordal if and only if $N(G)$ is totally balanced [Farber, 1984].
  - ▷ Totally balanced matrices are perfect [Conforti and Cornuéjols, 1995].

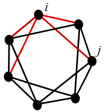

Chordal
$\mathcal{F}$
Strongly Chordal

A *web* W_n^r has $V(W_n^r) = \{1, \ldots, n\}$ and ij is an edge if i and j differ by at most r (mod n).



W_7^2

A *web* W_n^r has $V(W_n^r) = \{1, \ldots, n\}$ and ij is an edge if i and j differ by at most r (mod n).



W_7^2

- If $n \leq 2r+1$, $W_n^r = K_n$.
- For $s \geq 2$ the graph $W_{2s+1}^1 = C_{2s+1}$ and $W_{2s+1}^{s-1} = \overline{C_{2s+1}}$.

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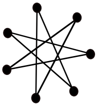

$\overline{W_7^2} = C_7$

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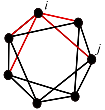

W_7^2

A *circulant matrix* C_n^r (r integer, $1 \leq r \leq n-1$) have columns in $\{1, \ldots, n\}$ and its rows are incidence vectors of $\{i+1, \ldots, i+r\}$ (+ mod n).

$$C_7^5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

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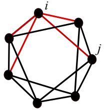

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- For $n > 2r+1$ and $r \geq 1$, $N(W_n^r) = C_n^{2r+1}$.
- C_n^{2r+1} is a clique-node matrix if and only if $n \geq 6r+1$, and $C_n^{2r+1} = \mathscr{C}(W_n^{2r})$.
- A non complete W_n^{2r} is perfect if and only if $n = 4r+2$ [Bianchi et al., 2016]

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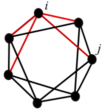

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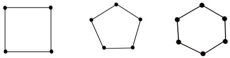
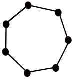
Complete graphs are the only web graphs in $\mathscr{F}$.

1. For $n \in \{4,5,6\}$, $C_n \notin \mathscr{F}$ $\rightarrow$ 

   since $N(C_n)$ is not an extended clique-node matrix

2. For $n \geq 7$, $C_n \notin \mathscr{F}$ $\rightarrow$  C_7

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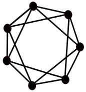
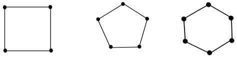


W_7^2

CYCLES DO NOT BELONG TO $\mathscr{F} = \{G : N(G)$ IS PERFECT$\}$

1. For $n \in \{4,5,6\}$, $C_n \notin \mathscr{F}$ $\rightarrow$ 

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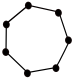
2. For $n \geq 7$, $C_n \notin \mathscr{F}$ $\rightarrow$  C_7

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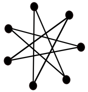
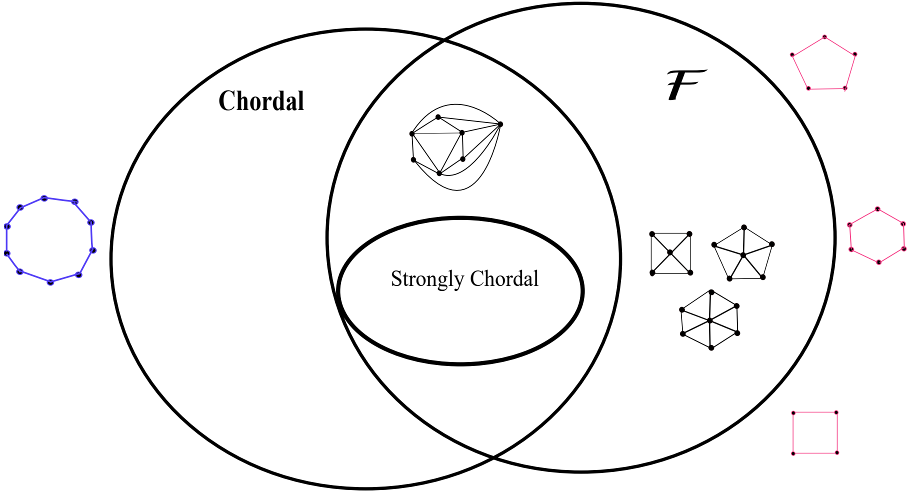
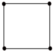


$\overline{W_7^2}$


Chordal
$\mathcal{F}$
Strongly Chordal

Characterize graphs G with extended clique-node $N(G)$

SOME SMALL GRAPHS WITH NON EXTENDED CLIQUE-NODE $N(G)$

- $G = C_4$  $N(C_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = J - I$

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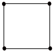
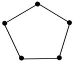
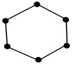
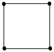
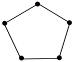
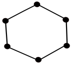
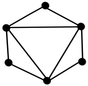
- $G = C_4$  $N(C_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = J - I$

- Similarly for $G = C_5$  and $G = C_6$ 

Some small graphs with non extended clique-node $N(G)$

- $G = C_4$  $N(C_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = J - I$

- Similarly for $G = C_5$  and $G = C_6$ 

- $G = S_3$  $N(S_3) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$

Theorem

Given a graph G and $\mathscr{P} = \{C_4, C_5, C_6, S_3\}$.
$N(G)$ is an extended clique-node matrix if and only if for every $\tilde{G} \subseteq G$ and $\tilde{G} \in \mathscr{P}$, there exists $v \in V(G) \setminus V(\tilde{G})$ s.t.

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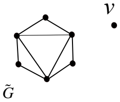

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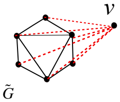

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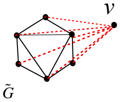

Proof.

$\Longrightarrow$ Previous analysis and C. and C.'s characterization.

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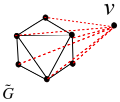


Proof.

$\Longleftarrow$ Suppose $N(G)$ contains a $(J-I)_{p\times p}$ as submatrix without the row of 1's.

It is enough to prove that the proper $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}_{3\times 3}$ forces G to have at least one induced subgraph in $\mathscr{P}$ and then use C. and C.'s th.

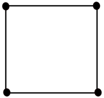
$J-I$ is a submatrix of a proper $G^* \subset G$ with $4 \leq |V(G^*)| \leq 6$.

SKETCH OF THE CHARACTERIZATION'S PROOF OF GRAPHS WITH EXTENDED CLIQUE-NODE $N(G)$.

▷ $|V(G^*)| = 4$

	1	2	3	4
2	0	1	1	1
1	1	0	1	1
4	1	1	0	1
3	1	1	1	0

$\Longrightarrow$



C_4

SKETCH OF THE CHARACTERIZATION'S PROOF OF GRAPHS WITH EXTENDED CLIQUE-NODE $N(G)$.

▷ $|V(G^*)| = 5$

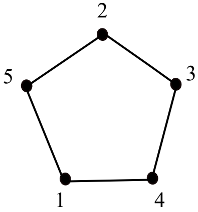
	1	2	3	4	5
2	0	1	1	0	1
4	1	0	1	1	α
5	1	1	0	α	1
1	1	0	β	1	1
3	β	1	1	1	0

SKETCH OF THE CHARACTERIZATION'S PROOF OF GRAPHS WITH EXTENDED CLIQUE-NODE $N(G)$.

▷ $|V(G^*)| = 5$

	1	2	3	4	5
2	0	1	1	0	1
4	1	0	1	1	0
5	1	1	0	0	1
1	1	0	0	1	1
3	0	1	1	1	0

$\Longrightarrow$



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	1	2	3	4	5
2	0	1	1	0	1
4	1	0	1	1	0
5	1	1	0	0	1
1	1	0	1	1	1
3	1	1	1	1	0

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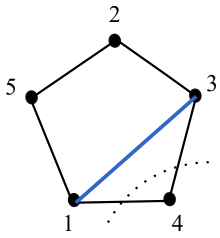

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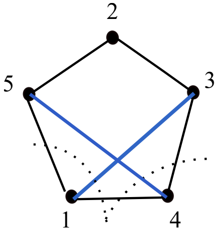


SKETCH OF THE CHARACTERIZATION'S PROOF OF GRAPHS WITH EXTENDED CLIQUE-NODE $N(G)$.

- $|V(G^*)| = 6$

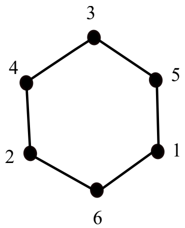
	1	2	3	4	5	6
4	0	1	1	1	α	β
5	1	0	1	α	1	λ
6	1	1	0	β	λ	1
1	1	μ	θ	0	1	1
2	μ	1	ω	1	0	1
3	θ	ω	1	1	1	0

Sketch of the characterization's proof of graphs with extended clique-node $N(G)$.

- $|V(G^*)| = 6$

	1	2	3	4	5	6
4	0	1	1	1	0	0
5	1	0	1	0	1	0
6	1	1	0	0	0	1
1	1	0	0	0	1	1
2	0	1	0	1	0	1
3	0	0	1	1	1	0

$\Longrightarrow$



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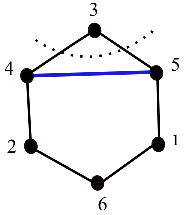
2	0	1	0	1	0	1
3	0	0	1	1	1	0

$\Longrightarrow$



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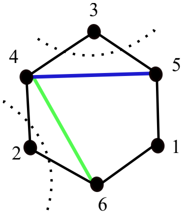

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- $|V(G^*)| = 6$

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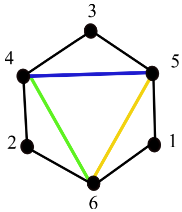
4	0	1	1	1	1	1
5	1	0	1	1	1	1
6	1	1	0	1	1	1
1	1	0	0	0	1	1
2	0	1	0	1	0	1
3	0	0	1	1	1	0

$\Longrightarrow$



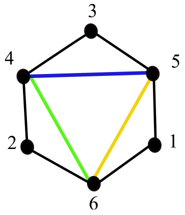
S_3

SKETCH OF THE CHARACTERIZATION'S PROOF OF GRAPHS WITH EXTENDED CLIQUE-NODE $N(G)$.

▷ $|V(G^*)| = 6$

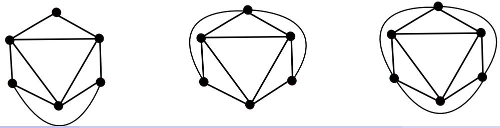
	1	2	3	4	5	6
4	0	1	1	1	1	1
5	1	0	1	1	1	1
6	1	1	0	1	1	1
1	1	0	0	0	1	1
2	0	1	0	1	0	1
3	0	0	1	1	1	0

$\Longrightarrow$



S_3

The remaining have induced C_4's:



$\mathscr{F} = \{G : N(G)$ IS PERFECT$\}$

Recall...

- We need two conditions for a graph to belong to $\mathscr{F}$.
- We have explored when $N(G)$ is an extended clique-node matrix.
- We will now focus on the perfection of the graph $G_Q(N(G))$.

PROPOSITION

If $G = (V, E)$ is such that:

- there is $G' \subset G$ whose associated clique graph G'_Q is not perfect (and thus $G' \notin \mathscr{F}$) and
- for all $v \in V(G')$, $N[v] \subset N[w]$ for some $w \in V \setminus V(G')$,

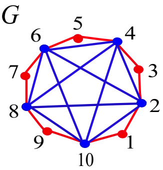
then the clique graph $G_Q(N(G))$ is not perfect, and thus $G \notin \mathscr{F}$.

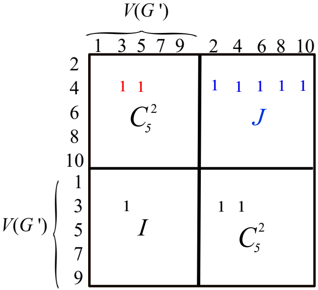
Recall that cycles $C_n \notin \mathscr{F}$ when $n \geq 7$, since $G_Q(N(C_n)) = W_n^2$ is not perfect.

But it is not enough to look at induced C_n's:

Example

$G=(V,E)$, $V=\{1,\ldots,4k+2\}$, even nodes form a clique and $N(i)=\{i-1,\ i+1\}$ for odd vertices i.

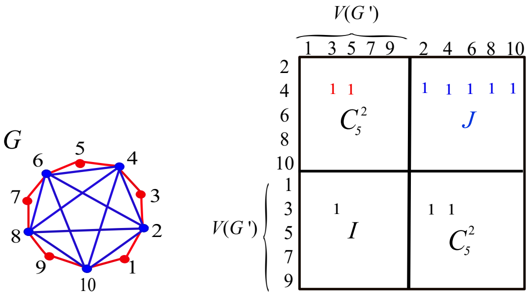




But it is not enough to look at induced C_n's:

EXAMPLE

$G = (V,E)$, $V = \{1,\ldots,4k+2\}$, even nodes form a clique and $N(i) = \{i-1,\ i+1\}$ for odd vertices i.

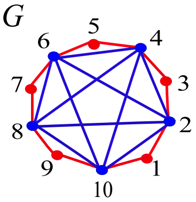


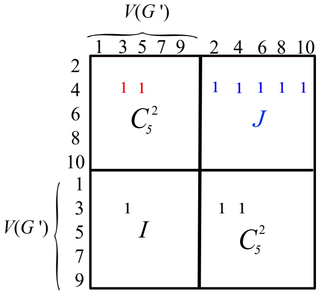
for all $v \in V(G')$, $N[v] \subset N[w]$ for some $w \in V \setminus V(G')$

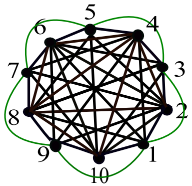
EXAMPLE (CONT.)

$G = (V,E)$, $V = \{1,\ldots,4k+2\}$, even nodes form a clique and $N(i) = \{i-1,\ i+1\}$ for odd vertices i.

- $G_Q(N(G))$ is the complete join of C_5 and K_5.
- $G_Q(N(G))$ is not perfect (it has the induced C_5, $G_Q(N(G)) \setminus \{2,4,6,8,10\}$.)

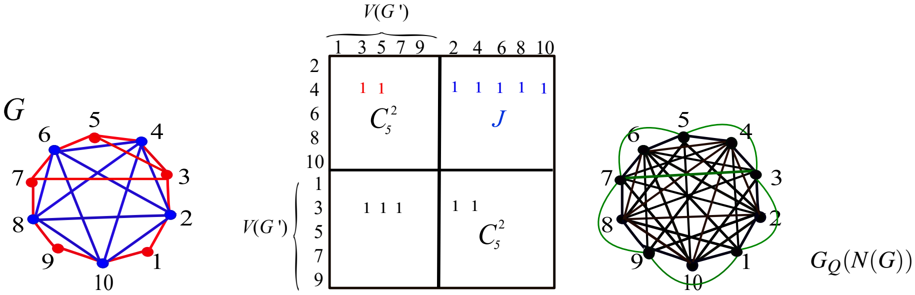


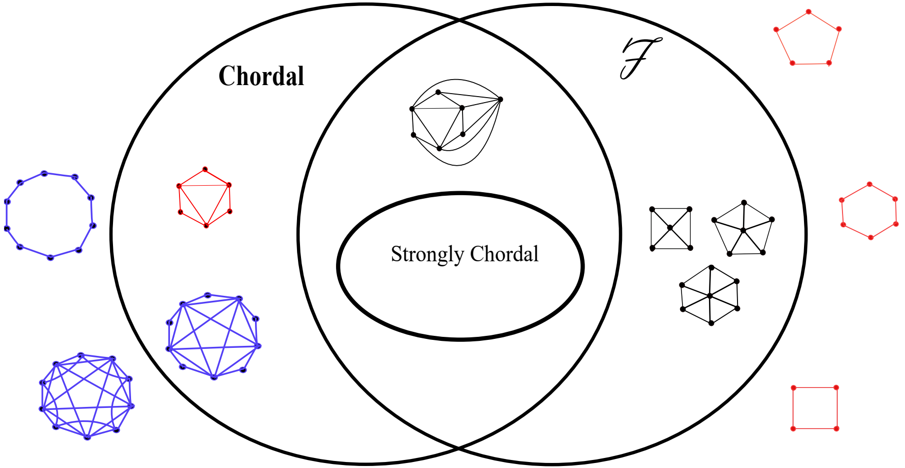




If the sufficient condition does not hold ?

For instance for $v = 3$, $N[v] \not\subset N[w]$ for all $w \in V \setminus V(G')$.




Chordal
$\mathcal{F}$
Strongly Chordal

1. From an ILP formulation of $\{k\}$PF, we found a sufficient condition for its optimality (the perfection of $N(G)$).
2. We began a structural study of graphs in $\mathscr{F}$ and present a characterization of graphs for which $N(G)$ is an extended clique-node matrix.
3. We gave neccesary conditions for a graph to belong to $\mathscr{F}$. We need to study in more depth the subgraphs G' of G not in $\mathscr{F}$ with non perfect G'_Q.

Open problems

1. Find neccesary conditions for a graph G to have $G_Q(N(G))$ perfect that are also suffcient conditions.
2. Find all minimal graphs that have non perfect $G_Q(N(G))$.
3. Find other type of characterizations of graphs with perfect $N(G)$.


Gracias