Spaces with non-integer dimension

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September 27, 2017

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Motivation

- Renormalization group and the ϵ expansion

Critical Exponents in 3.99 Dimensions*

Kenneth G. Wilson and Michael E. Fisher
Laboratory of Nuclear Studies and Baker Laboratory, Cornell University, Ithaca, New York 14850
(Received 11 October 1971)

Critical exponents are calculated for dimension $d=4-\epsilon$ with ϵ small, using renormalization-group techniques. To order ϵ the exponent γ is $1+\frac{1}{6}\epsilon$ for an Ising-like model and $1+\frac{1}{5}\epsilon$ for an XY model.

A generalized Ising model is solved here for dimension $d=4-\epsilon$ with ϵ small. Critical exponents[1] are obtained to order ϵ or ϵ^2. For $d>4$ the exponents are mean-field exponents[1] independent of ϵ; below $d=4$ the exponents vary continuously with ϵ. For example, the susceptibility exponent γ is $1+\frac{1}{6}\epsilon$ to order ϵ for $\epsilon>0$, and 1 exactly for $\epsilon<0$. The definitions for nonintegral d are trivial for the calculations reported here but may be more difficult for exact calculations to higher orders in ϵ. The exponents will be calculated using a recursion formula derived elsewhere[2] which represents critical behavior approximately in three dimensions but turns out to be exact to order ϵ (see the end of this paper). Exponents will also be obtained for the classical planar Heisenberg model (XY model) and a modified form of Baxter's eight-vertex model.[3]

VOLUME 28, NUMBER 4 PHYSICAL REVIEW LETTERS 24 JANUARY 1972

Motivation

- Dimensional regularization

IL NUOVO CIMENTO Vol. 12 B, N. 1 11 Novembre 1972

Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter.

C. G. Bollini and J. J. Giambiagi

Departamento de Física, Facultad de Ciencias Exactas
Universidad Nacional de La Plata

Summary. — We perform an analytic extension of quantum electrodynamics matrix elements as (analytic) functions of the number of dimensions of space (ν). The usual divergences appear as poles for ν integer. The renormalization of those matrix elements (for ν arbitrary) leads to expressions which are free of ultraviolet divergences for ν equal to 4. This shows that ν can be used as an analytic regularizing parameter with advantages over the usual analytic regularization method. In particular, gauge invariance is mantained for any ν.

Motivation

7.A.1

Nuclear Physics B44 (1972) 189–213. North-Holland Publishing Company

REGULARIZATION AND RENORMALIZATION OF GAUGE FIELDS

G. 't HOOFT and M. VELTMAN
Institute for Theoretical Physics *, *University of Utrecht*

Received 21 February 1972

Abstract: A new regularization and renormalization procedure is presented. It is particularly well suited for the treatment of gauge theories. The method works for theories that were known to be renormalizable as well as for Yang-Mills type theories. Overlapping divergencies are disentangled. The procedure respects unitarity, causality and allows shifts of integration variables. In non-anomalous cases also Ward identities are satisfied at all stages. It is transparent when anomalies, such as the Bell-Jackiw-Adler anomaly, may occur.

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- Basic idea

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- Dimension spectrum
  - Required for Generalization of local index theorem
  - Chern cocycle in De-Rahm cohomology $\leftrightarrow$ Local index (anomaly) of Dirac operator(canonical case)
  - Chern cocycle in Cyclic cohomology $\leftrightarrow$ Residue of zeta function for certain forms involving the Dirac operator in dimension spectrum poles

Non-commutative geometry

Geometric And Functional Analysis
Vol. 5, No. 2 (1995)

1016-443X/95/0200174-70$1.50+0.20/0

THE LOCAL INDEX FORMULA IN NONCOMMUTATIVE GEOMETRY

A. CONNES AND H. MOSCOVICI

We dedicate this paper to Misha Gromov

Abstract

In noncommutative geometry a geometric space is described from a spectral vantage point, as a triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of a $*$-algebra $\mathcal{A}$ represented in a Hilbert space $\mathcal{H}$ together with an unbounded selfadjoint operator D, with compact resolvent, which interacts with the algebra in a bounded fashion. This paper contributes to the advancement of this point of view in two significant ways: (1) by showing that any pseudogroup of transformations of a manifold gives rise to such a spectral triple of finite summability degree, and (2) by proving a general, in some sense universal, local index formula for arbitrary spectral triples of finite summability degree, in terms of the Dixmier trace and its residue-type extension.

Example of space with non-integer dimension[1]

- Spectral triple. Dirac operator
  - $\mathcal{A}$ is the commutative C^*-algebra of smooth functions over the n-dimensional torus T^n $n \in \mathbb{N}$.

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  - $\mathcal{H}$ is the Hilbert space of square integrable sections of a spinor bundle over T^n.
  - $D_\alpha : \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator given by,

$$D_\alpha = D(1 + D^2)^{-\alpha}, \ \alpha > 0$$

where D is the n-dimensional Dirac operator, i.e. $D = i\gamma_\mu \partial_\mu$

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Example of space with non-integer dimension

- Discrete dimension spectrum
  - Essentialy given by the poles of the zeta function for the Dirac operator given by,

$$\zeta_b^{D_\alpha}(z) = Tr[\pi(b)\,|D_\alpha|^{-z}]$$

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  - For this case the poles are located at,

$$z = \frac{n - 2k(1-\delta_{\alpha,0})}{1-2\alpha} \;\; , k = 0, 1, 2, \cdots$$

  which are interpreted as giving the dimensions of different parts of this space.

Differential algebra

- Differential,

$$\pi(\delta f) = df = [D_\alpha, f]$$

which leads to,

$$df = (1 + D^2)^{-\alpha}(Df) + [(1 + D^2)^{-\alpha} f - f(1 + D^2)^{-\alpha}] i\gamma \cdot \partial$$

which, for $\alpha \neq 0$, is a non-multiplicative operator.

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- Junk forms,

$$\pi(\omega) = 0 \;\; but \;\; \pi(\delta\omega) \neq 0$$

Example canonical triple($\alpha = 0$):

$$\omega = f\delta f - (\delta f)f \;\; , \delta\omega = \delta f \delta f$$

and,

$$\pi(\omega) = i\gamma_\mu (f(\partial_\mu f) - (\partial_\mu f) f) = 0$$

Differential algebra

but,

$$\pi(\delta\omega) = -\gamma_\mu \gamma_\nu \partial_\mu f \partial_\nu f = \partial_\mu f \partial_\nu f \neq 0$$

- No junk forms for $\alpha \neq 0$, non-multiplicativity of *df* plays an important role in this respect.

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- Forms of any order exists for $\alpha \neq 0$

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- No junk forms for $\alpha \neq 0$, non-multiplicativity of *df* plays an important role in this respect.
- Forms of any order exists for $\alpha \neq 0$
- For $\alpha \neq 0$ a vector has an infinite number of components.

The scalar field

- Action,

$$S = \frac{1}{2} < d\phi, d\phi >$$

where ϕ is a 0-form and the norm in the space forms is given by,

$$< \eta, \eta >= tr_\omega[\eta\eta^\dagger |D_\alpha|^{-d}]$$

tr_ω denotes Diximiers trace.

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- This can be expressed as a Wodzicki's residue,

$$tr_\omega(A) = \frac{1}{n(2\pi)^n} \int_{S^* T^n} tr\sigma^A_{-n}(x, \xi)$$

where $\sigma^A_{-n}(x, \xi)$ denotes the term of order $-n$ of the symbol of the operator A , (x, ξ) denote coordinates over the unit co-sphere on the cotangent bundle of T^n.

The scalar field

- Only the part of this space corresponding to the pole $z = \frac{n}{1-2\alpha}$ is considered. Leading to,

$$S = -\frac{2^{[\frac{n}{2}]} V_{S^{n-1}}}{n(2\pi)^n} \int_{T^n} \phi(D^2 + \frac{\alpha n}{1-2\alpha})(1+D^2)^{-2\alpha} \phi^*$$

  which is a non-local action in n dimensions.

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  which is a non-local action in n dimensions.

- The propagator for this theory is given by,

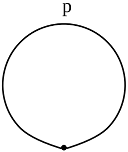
$$D(x-y) = \int d^n p \frac{(M^2+p^2)^{2\alpha}}{(p^2+m^2)} e^{-ip\cdot(x-y)}$$

  where units has been restored to the coordinates and ,

$$m^2 = M^2 \frac{\alpha n}{1-2\alpha}$$

One loop diagrams

- Tadpole,



$$I_\alpha^T(m) = \int d^n p \frac{(M^2 + p^2)^{2\alpha}}{(p^2 + m^2)}$$

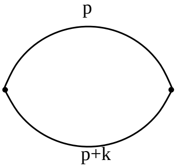
which leads to,

$$I_\alpha^T(m) = \frac{-\pi^{\frac{n}{2}}\Gamma(1-\frac{n}{2}-\alpha)}{\Gamma(-\alpha)\alpha}(M^2)^{\alpha+\frac{n}{2}-1}\,{}_2F_1(1, 1-\alpha-\frac{n}{2}, 1-\alpha, 1-\frac{m^2}{M^2})$$

For $0 \leq \alpha < 1/2$ the hypergeometric function divided by $\Gamma(-\alpha)\alpha$ present no poles. Singularities come from $\Gamma(1-\frac{n}{2}-\alpha)$ in the numerator, which coincide with the ones appearing in dimensional regularization.

One loop diagrams

- A loop involving two free propagators,



Same result as for the tadpole diagram. Singularities coincide with the ones appearing in dimensional regularization. Finite result for $0 < \alpha < 1/2$.

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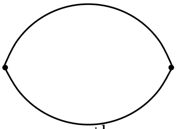

Same result as for the tadpole diagram. Singularities coincide with the ones appearing in dimensional regularization. Finite result for $0 < \alpha < 1/2$.

- The magic of analytic continuation. For $0 < \alpha < 1/2$ ultraviolet behaviour of propagator is worse than for $\alpha = 0$ however in that region there are no divergences. This is in sharp contrast with commonly employed regularizations that improve convergence by improving ultraviolet behaviour of propagators in momentum space.

Physical theories in non-integer dimensions. Unitariry and reflection positivity.[2]

- Is it possible to consider theories on non-integer dimensional spaces as physical theories?

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- Reflection positivity guanrantees the existence of a positive definite inner product on a Hilbert space when going to Minkowski space.

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- Working in Euclidean space this last question has an affirmative answer if the property of reflection positivity holds.
- Reflection positivity guanrantees the existence of a positive definite inner product on a Hilbert space when going to Minkowski space.
- Reflection positivity basics: Consider $\mathbb{R}_{d+1}$ and the plane $x_{d+1} = 0$, let $\mathbb{R}^+_{d+1}$ denote the points with $x_{d+1} > 0$ and θ the reflection on $x_{d+1} = 0$. The the requirement of reflection positivity for a theory with propagator $S(x - y)$ is,

$$(\theta f, f) \geq 0 \ \ , Support(f) \subset \mathbb{R}^+_{d+1}$$

where the scalar product $(,)$ is defined by,

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Physical theories in non-integer dimensions. Unitariry and reflection positivity.

$$(f, g) = \int d^{d+1}x\, d^{d+1}y\, \bar{f}(x) S(x-y) g(y)$$

and $\theta f = f(\theta x) \ \ , \theta x = (-x_{d+1}, \bar{x})$.

- It turns out that reflection positivity(RP) holds for $0 < \alpha < \frac{1}{n+2}$ and that RP does not hold for $\alpha < 0$.

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- It turns out that reflection positivity(RP) holds for $0 < \alpha < \frac{1}{n+2}$ and that RP does not hold for $\alpha < 0$.
- This result partially coincide with the ones of the so called conformal bootstrap. In this approach, non-integer dimension is incorporated by assuming that for non-integer dimension the antisymmetrization of objects with n-indices with $d < n$ does not vanish[3].

[3] Rychkov'15

Conclusions and outlook

- Comparison with dimensional regularization: There is no knowm Lagrangian formulation of the widely employed dimensional regularization technique. It is done separatedly for each integral appearing in a calculation of Feynman diagrams. The present approach can be viewed as a regularization scheme that is implementable at the Lagrangian level and presents the same singularity structure as dimensional regularization.

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- Comparison with non-commutative field theory(NCFT): In the present approach no non-commutativity of the coordinates is assumed. Non-commutativity enters at the level of the differential algebra through the deformed choice of the Dirac operator. Also, contributions are finite which is not what happens in NCCFT. Furthermore covariance is not spoiled as happens in NCFT's.

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- Physical theories in non-integer dimension: It has been shown that for the case of a scalar field reflection positivity holds and therefore for these theories the question of wether physical space has integer dimension or not can be properly addressed.

Conclusions and outlook

- Higher order forms theories: A very important feature of this approach is the fact that it is based on a well defined differential geometry. This allows to consider the generalization of any field theory defined in differential geometric terms to these deformed spaces. This includes gauge theories and gravity theories. Of course the resulting theories deserve to be studied in detail.

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- The ϵ-expansion and the conformal bootstrap: It is worth considering the corresponding computations in the present approach. In particular the isometries of these spaces play an important role in this respect. The NCG approach corresponds for these isometries to the computation of conserved quantities at the level of fields.