El problema inverso del cálculo de variaciones para sistemas discretos

Sebastián J. Ferraro

en colaboración con María Barbero-Liñán, Marta Farré Puiggalí y David Martín de Diego

Mar del Plata, 28 de Septiembre de 2017

Some standard notation

Q configuration space, $\dim Q = n$, with local coordinates (q^i).

$\tau_Q \colon TQ \to Q$, with local coordinates $(q^i, \dot{q}^i)$.

$\pi_Q \colon T^*Q \to Q$, with local coordinates (q^i, p_i).

We want to consider the inverse problem for:

- Continuous systems
  - Explicit
  - Implicit
  - Constrained
- Discrete systems
  - Explicit
  - Implicit
  - Constrained

Inverse problem — continuous setting

Classical inverse problem of the calculus of variations: determining whether a given system of explicit second order differential equations (SODE)

$$\ddot{q}^i = \Gamma^i(q^j, \dot{q}^j), \quad i, j = 1, \ldots, n$$

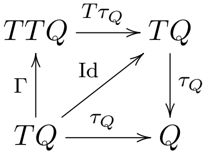
is equivalent (same solutions) to a system of Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0\,,$$

for a regular Lagrangian $L(q, \dot{q})$ to be determined. In that case we say that the SODE is *variational*.

A SODE Γ on TQ is a vector field $\Gamma \in \mathfrak{X}(TQ)$ such that

$$T\tau_Q(\Gamma(v_q)) = v_q \text{ for all } v_q \in T_qQ.$$



Locally, $\Gamma = \dot{q}^i \dfrac{\partial}{\partial q^i} + \Gamma^i(q, \dot{q}) \dfrac{\partial}{\partial \dot{q}^i}$, or $\Gamma(q^i, \dot{q}^i) = (q^i, \dot{q}^i, \dot{q}^i, \Gamma^i(q, \dot{q}))$.

The integral curves of Γ satisfy

$$\frac{dq^i}{dt} = \dot{q}^i, \quad \frac{d\dot{q}^i}{dt} = \Gamma^i(q, \dot{q}),$$

which is equivalent to

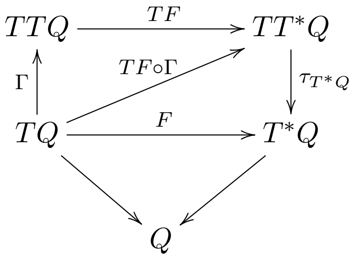
$$\frac{d^2q^i(t)}{dt^2} = \Gamma^i\left(q(t), \frac{dq(t)}{dt}\right).$$

Consider the canonical symplectic form $\omega_Q = dq^i \wedge dp_i$ on T^*Q.

The *tangent lift* of ω_Q to TT^*Q is a symplectic form

$$d_T\omega_Q = d\dot{q}^i \wedge dp_i + dq^i \wedge d\dot{p}_i \in \Omega^2(TT^*Q)$$

Theorem (Barbero-Liñán, Farré Puiggalí, Martín de Diego (2015))
A SODE Γ on TQ is variational if and only if there exists a local diffeomorphism $F\colon TQ \longrightarrow T^*Q$ of fibre bundles over Q such that $\mathrm{Im}(TF \circ \Gamma)$ is a Lagrangian submanifold of the symplectic manifold $(TT^*Q, d_T\omega_Q)$.



The continuous *implicit* case

- The explicit SODE on TQ is replaced by a submanifold of TTQ.
- $T^{(2)}Q$: the second order tangent bundle of Q.

$$T^{(2)}Q = \{v \in TTQ : T\tau_Q(v) = \tau_{TQ}(v)\} \subset TTQ \,,$$

  which locally is the set of $(q, v, \dot{q}, \dot{v}) \in TTQ$ such that $v = \dot{q}$.

  Local coordinates on $T^{(2)}Q$: $(q, \dot{q}, \ddot{q})$.

- Implicit system of second order differential equations:

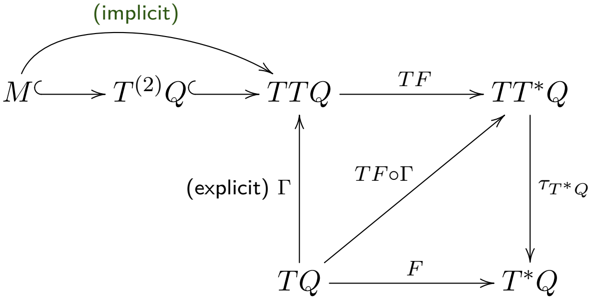
$$\Phi^i(q, \dot{q}, \ddot{q}) = 0, \quad i = 1, \ldots, n \,,$$

  such that $C := \left(\dfrac{\partial \Phi}{\partial \ddot{q}}\right)$ is regular. This defines $M \subset T^{(2)}Q \subset TTQ$.

- As before, we call the system $\Phi(q, \dot{q}, \ddot{q}) = 0$ *variational* if it is equivalent to the Euler–Lagrange equations for a regular Lagrangian $L\colon TQ \to \mathbb{R}$ (same solutions).

Theorem (~~Explicit~~ Implicit continuous case)

~~A SODE Γ on TQ~~ $\Phi(q, \dot{q}, \ddot{q}) = 0$ (as above) is variational if and only if there exists a local diffeomorphism $F\colon TQ \longrightarrow T^*Q$ of fibre bundles over Q such that ~~$\mathrm{Im}(TF \circ \Gamma)$~~ TF(M) is a Lagrangian submanifold of the symplectic manifold $(TT^*Q, d_T\omega_Q)$.



This leads to the following Helmholtz conditions

$$\frac{\partial F_i}{\partial \dot{q}^j} = \frac{\partial F_j}{\partial \dot{q}^i} \,,$$

$$\frac{\partial^2 F_i}{\partial \dot{q}^j \partial q^k}\dot{q}^k + \frac{\partial F_i}{\partial q^j} + \frac{\partial^2 F_i}{\partial \dot{q}^j \partial \dot{q}^k}\ddot{q}^k - \frac{\partial F_j}{\partial q^i} = \frac{\partial F_i}{\partial \dot{q}^k}\frac{\partial \Phi^r}{\partial \dot{q}^j}(C^{-1})^k_r \,,$$

$$\frac{\partial^2 F_i}{\partial q^j \partial q^k}\dot{q}^k + \frac{\partial^2 F_i}{\partial q^j \partial \dot{q}^k}\ddot{q}^k - \frac{\partial F_i}{\partial \dot{q}^k}\frac{\partial \Phi^r}{\partial q^j}(C^{-1})^k_r = \text{same with } i \leftrightarrows j.$$

That is, the system $\Phi(q, \dot{q}, \ddot{q}) = 0$ is variational if and only if there is a solution $F \colon TQ \to T^*Q$ (local diffeomorphism over Q) to the above Helmholtz conditions.

If the system is variational with Lagrangian L, then one can take

$$F := \mathbb{F}L \colon TQ \to T^*Q$$

$$(q, \dot{q}) \mapsto \left(q, \frac{\partial L}{\partial \dot{q}}\right)$$

Discrete mechanics

We want to do something similar for discrete systems.

Introduction to discrete mechanics

- $Q \times Q$: discrete version of TQ.
- Instead of curves on Q, take sequences of points $q_0, q_1, \ldots, q_N$.
- Instead of a function $L\colon TQ \to \mathbb{R}$, we consider a *discrete Lagrangian*

$$L_d\colon Q \times Q \to \mathbb{R}.$$

- Discrete action:

$$S_d(q_0, \ldots, q_N) = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$

Taking variations of the sequence with fixed endpoints q_0 and q_N and extremizing the discrete action we obtain the *discrete Euler-Lagrange equations* (or DEL equations)

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0 \quad \text{for all} \quad k = 1, \ldots, N-1\,,$$

where $D_1 L_d(q_{k-1}, q_k) \in T^*_{q_{k-1}} Q$ and $D_2 L_d(q_{k-1}, q_k) \in T^*_{q_k} Q$.

Assume that L_d is regular, that is, $D_{12} L_d$ is a regular matrix. Then we can solve for q_{k+1} and obtain an evolution map (discrete flow)

$$\begin{array}{rccc} \Phi_{L_d}: & Q \times Q & \longrightarrow & Q \times Q \\ & (q_{k-1}, q_k) & \longmapsto & (q_k, q_{k+1}(q_{k-1}, q_k))\,, \end{array}$$

Define the two discrete Legendre transformations:

$$\mathbb{F}^+L_d,\, \mathbb{F}^-L_d : Q \times Q \longrightarrow T^*Q\,,$$

$$\begin{aligned}
\mathbb{F}^+L_d(q_{k-1}, q_k) &= (q_k, D_2L_d(q_{k-1}, q_k)) \in T^*_{q_k}Q\,,\\
\mathbb{F}^-L_d(q_{k-1}, q_k) &= (q_{k-1}, -D_1L_d(q_{k-1}, q_k)) \in T^*_{q_{k-1}}Q\,.
\end{aligned}$$

We can pull back the canonical symplectic form by either Legendre transformation and get the same result:

$$(\mathbb{F}^+L_d)^*\omega_Q = (\mathbb{F}^-L_d)^*\omega_Q =: \Omega_{L_d} \;\in\; \Omega^2(Q \times Q)$$

Locally,

$$\Omega_{L_d}(q_{k-1}, q_k) = -\frac{\partial^2 L_d}{\partial q^i_{k-1}\partial q^j_k} dq^i_{k-1} \wedge dq^j_k\,.$$

It can be shown that the discrete flow $\Phi_{L_d}\colon (q_{k-1}, q_k) \mapsto (q_k, q_{k+1}(q_{k-1}, q_k))$ preserves Ω_{L_d}.

Inverse problem — discrete setting

First situation: explicit case

Consider $Q \times Q \times Q \times Q$ as a discrete version of TTQ.

The discrete second order submanifold is defined by

$$\ddot{Q}_d = \{(q, \bar{q}, \bar{q}, \hat{q}) \in Q^4\} \cong Q \times Q \times Q$$

A second order difference equation (SOdE) is (by def.) a map of the form

$$\begin{array}{rccc} \Gamma : & Q \times Q & \longrightarrow & Q \times Q \times Q \times Q \\ & (q_{k-1}, q_k) & \longmapsto & (q_{k-1}, q_k, q_k, \tilde{\Gamma}(q_{k-1}, q_k)) \,, \end{array}$$

(discrete analogue of $\Gamma \colon TQ \to TTQ$).

Note that Γ takes values on $\ddot{Q}_d \subset Q^4$ (a *continuous* SODE takes values on $T^{(2)}Q \subset TTQ$).

We will sometimes say "the SOdE $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$".

Definition

The explicit second order difference equation $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$ is *variational* if and only if there is a regular discrete Lagrangian $L_d : Q \times Q \longrightarrow \mathbb{R}$ such that

$$q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k) \quad \text{and} \quad D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0$$

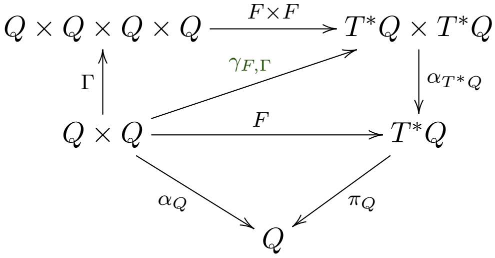
admit the same solutions.

Let $\alpha_Q : Q \times Q \longrightarrow Q$ be the projection onto the first factor. It plays the role of $\tau_Q : TQ \longrightarrow Q$.

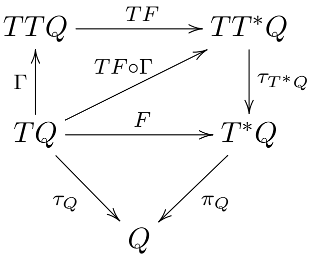
Similarly, $\alpha_{T^*Q} : T^*Q \times T^*Q \longrightarrow T^*Q$ is the projection onto the first factor. It plays the role of $\tau_{T^*Q} : TT^*Q \longrightarrow T^*Q$.

For a given SOdE $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$ and a local diffeomorphism $F : Q \times Q \longrightarrow T^*Q$ over the identity, we define $\gamma_{F,\Gamma} := (F \times F) \circ \Gamma$

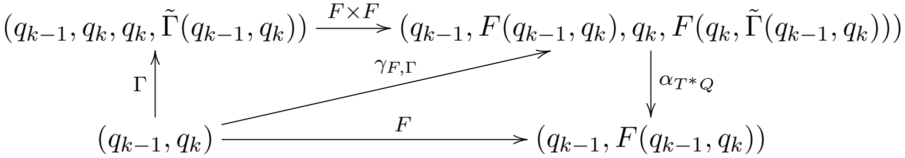
Discrete



Continuous



For $(q_{k-1}, q_k) \in Q \times Q$ the diagram is the following:



Consider on $T^*Q \times T^*Q$ the symplectic form $\Omega_Q := \beta^*_{T^*Q}\omega_Q - \alpha^*_{T^*Q}\omega_Q$. In coordinates $(q_{k-1}, p_{k-1}, q_k, p_k)$,

$$\Omega_Q = dq^i_k \wedge dp_{k,i} - dq^i_{k-1} \wedge dp_{k-1,i}$$

Theorem

The second order difference equation $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$ is variational if and only if there is a local diffeomorphism $F : Q \times Q \longrightarrow T^*Q$ over the identity such that $\mathrm{Im}(\gamma_{F,\Gamma})$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$.

Proof. Assume there is an F as in the statement, so $\mathrm{Im}(\gamma_{F,\Gamma})$ is a Lagrangian submanifold. This means:

- $\mathrm{Im}(\gamma_{F,\Gamma})$ is a submanifold of half the dimension of $T^*Q \times T^*Q$,
- Isotropy condition: $\gamma^*_{F,\Gamma}\Omega_Q = 0$ holds.

Recall that $\omega_Q = -d\theta_Q \in \Omega^2(T^*Q)$, where θ_Q is the canonical one-form on T^*Q. Pullbacks commute with d, so

$$\begin{aligned} 0 = \gamma^*_{F,\Gamma}\Omega_Q &= \gamma^*_{F,\Gamma}(\beta^*_{T^*Q}\omega_Q - \alpha^*_{T^*Q}\omega_Q) \\ &= -d\Big[(\beta_{T^*Q} \circ \gamma_{F,\Gamma})^*\theta_Q - (\alpha_{T^*Q} \circ \gamma_{F,\Gamma})^*\theta_Q\Big] \quad \text{closed} \end{aligned}$$

By the Poincaré Lemma, there is a (local) $L_d : Q \times Q \longrightarrow \mathbb{R}$ such that

$$(\beta_{T^*Q} \circ \gamma_{F,\Gamma})^*\theta_Q - (\alpha_{T^*Q} \circ \gamma_{F,\Gamma})^*\theta_Q = dL_d$$

In local coordinates, this becomes

$$\begin{aligned} &- F_i(q_{k-1}, q_k)dq^i_{k-1} + F_i(q_k, \tilde{\Gamma}(q_{k-1}, q_k))dq^i_k = \\ &\qquad \frac{\partial L_d}{\partial q^i_{k-1}}(q_{k-1}, q_k)dq^i_{k-1} + \frac{\partial L_d}{\partial q^i_k}(q_{k-1}, q_k)dq^i_k \ , \end{aligned}$$

that is,

$$\begin{aligned} D_1 L_d(q_{k-1}, q_k) &= -F(q_{k-1}, q_k) \\ D_2 L_d(q_{k-1}, q_k) &= F(q_k, \tilde{\Gamma}(q_{k-1}, q_k)). \end{aligned}$$

So far we have:

- We assumed that for the system $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$ (1), there is an F as in the statement.

- We found $L_d \colon Q \times Q \to \mathbb{R}$ using the Poincaré Lemma, and it satisfies

$$D_1 L_d(q_{k-1}, q_k) = -F(q_{k-1}, q_k) \tag{2}$$

$$D_2 L_d(q_{k-1}, q_k) = F(q_k, \tilde{\Gamma}(q_{k-1}, q_k)), \tag{3}$$

  for all (q_{k-1}, q_k) (locally).

Equation (2) means that $F = \mathbb{F}^- L_d$. Also, F being an diffeomorphism implies $D_{12} L_d$ regular, so L_d is regular.

$$-D_1 L_d(q_k, q_{k+1}) \overset{(2)}{=} F(q_k, q_{k+1}) \overset{(1)}{=} F(q_k, \tilde{\Gamma}(q_{k-1}, q_k)) \overset{(3)}{=} D_2 L_d(q_{k-1}, q_k)$$

which shows that $(q_{k-1}, q_k, q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k))$ satisfies the discrete Euler-Lagrange equations for L_d.

For the converse, assume now that $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$ is variational, with regular discrete Lagrangian L_d.

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0 \overset{\text{variationality}}{\iff} q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$$
$$\overset{D_{12}L_d \text{ regular}}{\iff} D_1 L_d(q_k, q_{k+1}) = D_1 L_d(q_k, \tilde{\Gamma}(q_{k-1}, q_k))$$

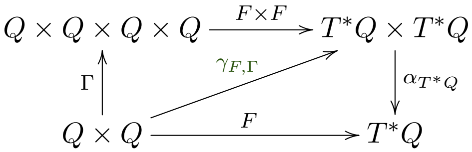
Take (q_{k-1}, q_k, q_{k+1}) satisfying the DEL equations. This implies

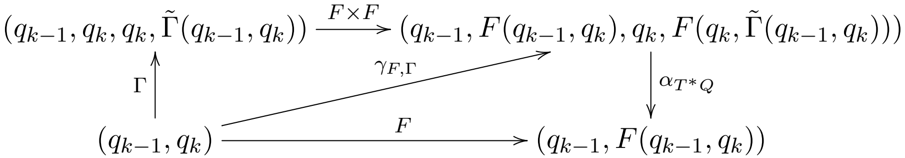
$$D_2 L_d(q_{k-1}, q_k) = -D_1 L_d(q_k, \tilde{\Gamma}(q_{k-1}, q_k))\,.$$

Define $F\colon Q \times Q \to T^*Q$ as

$$F(q_{k-1}, q_k) = -D_1 L_d(q_{k-1}, q_k).$$

Let us check that $\mathrm{Im}(\gamma_{F,\Gamma}) = \mathrm{Im}((F \times F) \circ \Gamma)$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$.





Then $\mathrm{Im}(\gamma_{F,\Gamma})$ is given by

$$\Big(q_{k-1}, -D_1 L_d(q_{k-1}, q_k), q_k, -D_1 L_d(q_k, \tilde{\Gamma}(q_{k-1}, q_k))\Big) = \\ (q_{k-1}, -D_1 L_d(q_{k-1}, q_k), q_k, D_2 L_d(q_{k-1}, q_k))\,.$$

$\mathrm{Im}(\gamma_{F,\Gamma})$ is the set of elements of $T^*Q \times T^*Q$ of the form

$$(q_{k-1}, -D_1 L_d(q_{k-1}, q_k), q_k, D_2 L_d(q_{k-1}, q_k))$$

This very similar to the image of dL_d. In fact,

$$\begin{aligned} dL_d(q_{k-1}, q_k) &= D_1 L_d(q_{k-1}, q_k) dq_{k-1} + D_2 L_d(q_{k-1}, q_k) dq_k \\ &\equiv (q_{k-1}, q_k, D_1 L_d(q_{k-1}, q_k), D_2 L_d(q_{k-1}, q_k)) \in T^*(Q \times Q). \end{aligned}$$

$\mathrm{Im}\, dL_d$ is a Lagrangian submanifold of $(T^*(Q \times Q), \omega_{Q\times Q})$ with the canonical symplectic structure. (The image of a closed 1-form is a Lagrangian submanifold.)

Using the symplectomorphism

$$\begin{array}{rccc} \Psi : & (T^*(Q \times Q), \omega_{Q\times Q}) & \longrightarrow & (T^*Q \times T^*Q, \Omega_Q) \\ & (\alpha_{q_0}, \alpha_{q_1}) & \longmapsto & (-\alpha_{q_0}, \alpha_{q_1}) \end{array}$$

we get that $\mathrm{Im}(\gamma_{F,\Gamma})$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$. $\square$

Discrete Helmholtz conditions

If we impose that $\mathrm{Im}(\gamma_{F,\Gamma})$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$ for a given SOdE Γ then we get the following conditions on F:

$$\frac{\partial F_i}{\partial Q_1^j}(q_{k-1}, q_k) = \frac{\partial F_j}{\partial Q_1^i}(q_{k-1}, q_k)\,,$$

$$\frac{\partial F_i}{\partial Q_2^l}(q_k, \tilde{\Gamma}(q_{k-1}, q_k))\frac{\partial \tilde{\Gamma}^l}{\partial q_k^j} = \frac{\partial F_j}{\partial Q_2^l}(q_k, \tilde{\Gamma}(q_{k-1}, q_k))\frac{\partial \tilde{\Gamma}^l}{\partial q_k^i}$$

$$\frac{\partial F_i}{\partial Q_2^j}(q_{k-1}, q_k) + \frac{\partial F_j}{\partial Q_2^l}(q_k, \tilde{\Gamma}(q_{k-1}, q_k))\frac{\partial \tilde{\Gamma}^l}{\partial q_{k-1}^i} = 0\,,$$

where $\partial/\partial Q_1$, $\partial/\partial Q_2$ denote partial derivatives with respect to the first and second slot respectively.

The system $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$ is variational if and only if there is a local diffeomorphism $F\colon Q \times Q \to T^*Q$ satisfying the discrete Helmholtz conditions.

Second situation: implicit case

Here a system of second order difference equations is given by a submanifold $M \subset Q \times Q \times Q$. We assume that M is given by

$$\Phi^i(q_{k-1}, q_k, q_{k+1}) = 0, \quad i = 1, \ldots, n,$$

such that $C := \left(\frac{\partial \Phi}{\partial q_{k+1}}\right)$ is invertible.

Definition

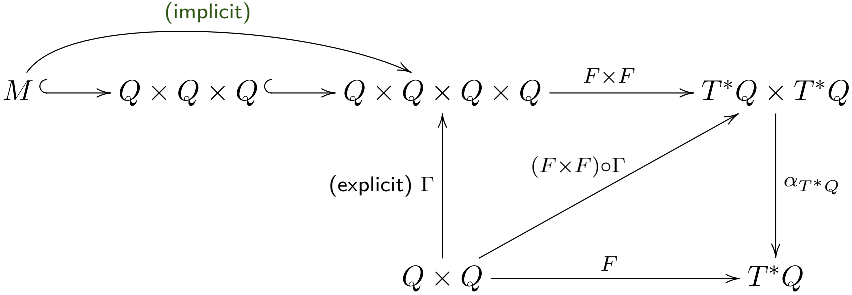
The implicit system of second order difference equations $\Phi^i(q_{k-1}, q_k, q_{k+1}) = 0$, $i = 1, \ldots, n$, is *variational* if and only if there is a regular discrete Lagrangian $L_d : Q \times Q \longrightarrow \mathbb{R}$ such that both systems

$$\Phi(q_{k-1}, q_k, q_{k+1}) = 0 \quad \text{and} \quad D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0$$

admit the same solutions.

Proposition

The ~~explicit SOdE $q_{k+1} = \tilde{\Gamma}(q_{k-1}, q_k)$~~ implicit SOdE $M \subset Q \times Q \times Q$ defined by $\Phi(q_{k-1}, q_k, q_{k+1}) = 0$ is variational if and only if there is a local diffeomorphism $F : Q \times Q \longrightarrow T^*Q$ over the identity such that ~~$\mathrm{Im}(\gamma_{F,\Gamma})$~~ $\mathrm{Im}((F \times F)|_M)$ is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$.



We get the **implicit discrete Helmholtz conditions**

$$\frac{\partial F_i}{\partial Q_1^j}(q_{k-1}, q_k) = \frac{\partial F_j}{\partial Q_1^i}(q_{k-1}, q_k)\,,$$

$$\frac{\partial F_j}{\partial Q_1^i}(q_k, q_{k+1}) - \frac{\partial F_j}{\partial Q_2^l}(q_k, q_{k+1})\frac{\partial \Phi^r}{\partial q_k^i}(C^{-1})_r^l = \text{same with } i \leftrightarrows j\,,$$

$$\frac{\partial F_i}{\partial Q_2^j}(q_{k-1}, q_k) = \frac{\partial F_j}{\partial Q_2^l}(q_k, q_{k+1})\frac{\partial \Phi^r}{\partial q_{k-1}^i}(C^{-1})_r^l$$

Variationality of continuous constrained systems

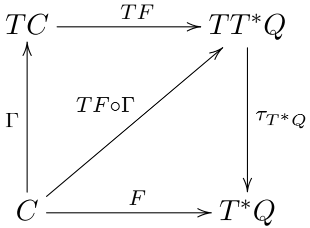
Let $C \subset TQ$ be a submanifold that can be locally described by

$$\underbrace{\dot{q}^\alpha}_{\text{“dependent” velocities}} = \psi^\alpha(q^i, \overbrace{\dot{q}^a}^{\text{“free” velocities}})$$

Let $\Gamma \in \mathfrak{X}(C)$ be a SODE. Locally,

$$\ddot{q}^a = \Gamma^a(q^i, \dot{q}^a)$$
$$\dot{q}^\alpha = \psi^\alpha(q^i, \dot{q}^a).$$

Definition (Barbero-Liñán, Farré Puiggalí, Martín de Diego (2015))
A SODE Γ on $C \subset TQ$ is variational if there exists an immersion $F\colon C \to T^*Q$ over Q such that $\mathrm{Im}(TF \circ \Gamma)$ is an isotropic manifold of $(TT^*Q, d_T\omega_Q)$.



Theorem (Barbero-Liñán, Farré Puiggalí, Martín de Diego (2015))
If a SODE Γ on C is variational, then there exists a Lagrangian $L\colon TQ \to \mathbb{R}$ such that the integral curves of Γ are those solutions $(q(t), \dot{q}(t))$ of the Euler–Lagrange equations that stay on C.

Variationality of discrete constrained systems

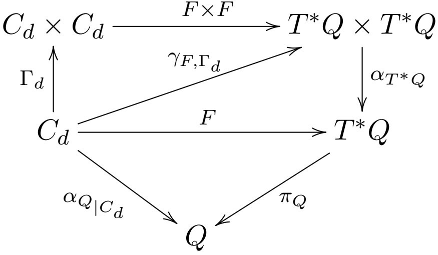
Let $C_d \subset Q \times Q$ be a submanifold defined by the discrete constraints $q_k^\alpha = \psi^\alpha(q_{k-1}^i, q_k^a)$

$$\underbrace{q_k^\alpha}_{\text{"dependent" components}} = \psi^\alpha(q_{k-1}^i, \overbrace{q_k^a}^{\text{"free" components}})$$

Let $\Gamma_d : C_d \longrightarrow C_d \times C_d$ be an explicit second order difference equation on C_d, that is,

$$\Gamma_d \overbrace{(q_{k-1}, q_k)}^{\in C_d} = (\overbrace{q_{k-1}, q_k,}^{\in C_d} \overbrace{q_k, \tilde{\Gamma}_d(q_{k-1}, q_k)}^{\in C_d}) \in \ddot{Q}_d$$

Given an immersion $F : C_d \longrightarrow T^*Q$ we define $\gamma_{F,\Gamma} := (F \times F) \circ \Gamma_d$, as shown in the following commutative diagram:



Definition

A SOdE Γ_d on C_d is variational if there exists an immersion $F : C_d \longrightarrow T^*Q$ such that $\mathrm{Im}(\gamma_{F,\Gamma_d})$ is an isotropic submanifold of $(T^*Q \times T^*Q, \Omega_Q)$.

As before, this leads to discrete Helmholtz conditions for constrained systems.

The meaning of variationality for constrained systems

If the constrained SOdE Γ_d on C_d is variational, then we have an isotropic submanifold $\mathrm{Im}(\gamma_{F,\Gamma_d})$ of $(T^*Q \times T^*Q, \Omega_Q)$. It can be extended to a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_Q)$ and it is possible to find a discrete Lagrangian $L_d\colon Q \times Q \to \mathbb{R}$ such that its (free) dynamics, when restricted to C_d, coincide with the original SOdE Γ_d.

Some natural questions:

1. Given a continuous variational SODE Γ on a submanifold $C \subset TQ$, find integrators Γ_d that are also variational.

2. From the existing integrators for nonholonomic systems, detect the ones that preserve the variational property.

Example

Discrete Lagrange-d'Alembert (DLA) algorithm (Cortés and Martínez, 2001).

Given a nonholonomic system, that is, a Lagrangian $L : TQ \to \mathbb{R}$ and a nonintegrable distribution $D \subset TQ$, choose a discrete Lagrangian L_d and a discrete constraint space $D_q \subset Q \times Q$.

The DLA integrator is then

$$\begin{aligned} D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) &= \lambda_a w^a(q_k)\,, \\ w_d^a(q_k, q_{k+1}) &= 0\,, \end{aligned}$$

where λ_a are Lagrange multipliers, w^a are the constraint one-forms, and w_d^a are a discretization of the contraint one-forms.

Particular example: the vertical rolling disk.

The continuous constrained system is variational: for example, two alternative immersions $F\colon C \to T^*Q$ are

$$F_1(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}) = (\theta, \varphi, x, y, 2\dot{\theta}, \dot{\varphi}, 0, 0),$$

$$F_2(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}) = \left(\theta, \varphi, x, y, \frac{\dot{\theta}}{\dot{\varphi}}, \dot{\varphi} - \frac{\dot{\theta}^2}{2\dot{\varphi}^2}\left(1 + \cos(\varphi) + \sin(\varphi)\right), \frac{\dot{\theta}}{\dot{\varphi}}, \frac{\dot{\theta}}{\dot{\varphi}}\right)$$

Together with the constrained SODE Γ (continuous) for the rolling disk, both provide isotropic submanifolds of TT^*Q.

On the discrete side, we consider the DLA equations with midpoint discretizations for the constraints (L does not depend on the positions).

We might ask whether the discretizations of F_1 and F_2 could serve to show that the discrete system is variational.

Define $F_{1d} : C_d \longrightarrow T^*Q$ as a discretization of F_1:

$$F_{d1}(\theta_{k-1}, \varphi_{k-1}, x_{k-1}, y_{k-1}, \theta_k, \varphi_k) = \Big(\theta_{k-1}, \varphi_{k-1}, x_{k-1}, y_{k-1},$$
$$2\frac{\theta_k - \theta_{k-1}}{h}, \frac{\varphi_k - \varphi_{k-1}}{h}, 0, 0\Big)$$

This gives rise to an isotropic submanifold of $T^*Q \times T^*Q$, so the discrete constrained system is variational.

However, if we take a discretization of F_2, it does *not* give rise to an isotropic submanifold.

Now change the discretization from midpoint to initial point (for the DLA only; F_1 does not depend on the positions). Then F_{1d} does not give an isotropic submanifold.

¡Gracias!