On Marsden-Ratiu Reduction

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Contents

Reference of the main article on the subject:

Marsden, J.E. and Ratiu, T.S. [1986], Reduction of Poisson manifolds. Lett. Math. Phys., 11, pag 161 to 169

Basic notation

M manifold, $N \subseteq M$ submanifold, $\varphi : N \to M$ inclusion,

$B \subseteq TM|N$ vector subbundle,

$F = TN \cap B$.

Hypothesis. F is an integrable regular distribution which defines a surjective submersion onto orbit space

$$\nu : N \to \bar{N}. \tag{1}$$

Define the space of B**-invariant functions** as follows

$$C^\infty(M)^B = \{f \in C^\infty(M) : df|B = 0\}. \tag{2}$$

Observation
(a) $\nu^* : C^\infty(\bar{N}) \to C^\infty(N)^F$ is a ring isomorphism

(b) If $B = 0$ then $C^\infty(M)^B = C^\infty(M)$; $C^\infty(\bar{N}) = C^\infty(N)^F$

(c) If $M = N$ then $B = F$ and $\nu^* : C^\infty(\bar{M}) \to C^\infty(M)^B$

(d) One can prove that $\varphi^* : C^\infty(\bar{M})^B \to C^\infty(M)^F$ is a surjective ring morphism.

Example. Symplectic reduction can be studied from this perspective, as we will show next.

(P, ω) symplectic manifold

G acts on P freely and properly by symplectomorphisms

$J : P \to \mathfrak{g}^*$ equivariant momentum map, μ regular value of J

G_μ isotropy subgroup acts freely and properly on $J^{-1}(\mu)$

Assume that G and G_μ are connected, for simplicity

Now let us interpret M, N, φ, ν as follows

$M = P$

$N = J^{-1}(\mu)$

$\varphi = i_\mu$, $i_\mu : J^{-1}(\mu) \to P$ inclusion

Define $B \subseteq TP|J^{-1}(\mu)$ by $B_x := T_x Gx$, for $x \in J^{-1}(\mu)$

Since $F = TJ^{-1}(\mu) \cap B$ one can prove that $F_x = T_x\left(G_\mu x\right)$

$C^\infty(J^{-1}(\mu))^{G_\mu} = C^\infty(J^{-1}(\mu))^F$, that is, G_μ-invariance $\equiv$ F-invariance

(P_μ, ω_μ), reduced symplectic space, $P_\mu = \bar{N}$

If f, g are B-invariant, that is

$$f, g \in C^\infty(P)^B \tag{3}$$

then $f|J^{-1}(\mu), g|J^{-1}(\mu)$ are F-invariant, that is

$$f|J^{-1}(\mu), g|J^{-1}(\mu) \in C^\infty(J^{-1}(\mu))^F. \tag{4}$$

and $f|J^{-1}(\mu), g|J^{-1}(\mu)$ pass to the quotient $f_\mu, g_\mu \in C^\infty(P_\mu)$.

Using the Marsden-Weinstein (symplectic) reduction the following formula obtains

$$\{f, g\}_P(x) = \{f_\mu, g_\mu\}_{P_\mu}(p_\mu(x)), \tag{5}$$

for all $x \in J^{-1}(\mu)$.

Is it possible to obtain a similar formula in the general case?

Main Result of Marsden-Ratiu Reduction We shall use the notation introduced before. In addition, we assume that there is a Poisson structure $\{,\}$ on M.

By definition $B \subseteq TM|N$ is canonical if the following condition is satiesfied,

$f, g \in C^\infty(M)^B$ implies that $\{f, g\}_M \in C^\infty(M)^B$.

For instance, the bundle B defined in the example of symplectic reduction is canonical.

In general, if B is canonical one can construct a bracket on $\bar{N}$ as follows.

First, for given $\bar{f}, \bar{g} \in C^\infty(\bar{N})$ there are uniquely determined $f, g \in C^\infty(N)^F$ defined by $f = \nu^* \bar{f}$, $g = \nu^* \bar{g}$.

Choose $f^B, g^B \in C^\infty(M)^B$ using the surjectivity of φ^*. The main result in Marsden-Ratiu reduction is the following.

Theorem. The expression $\{\bar{f}, \bar{g}\}_{\bar{N}}(\nu(x)) = \{f^B, g^B\}_M(x)$ gives a well defined bracket on $\bar{N}$ if and only if the following M-R condition is satisfied

$$\sharp(B^\circ) \subseteq TN + B \tag{6}$$

Observation.
(a) If $B = 0$ the M-R condition is equivalent to $\sharp(B^\circ) \subseteq TN$, that is, N is a Poisson submanifold of M. In this special case $C^\infty(M)^B = C^\infty(M)$ and N is isomorphic to $\bar{N}$.

(b) In the article:

Falceto, F. and Zambon, M. [2008], An extension of the Marsden-Ratiu reduction for Poisson manifolds, Lett. Math. Phys. 85, pag. 203 to 219,

it has been proven that if B is canonical and $B \neq 0$ then $\sharp(B^\circ) \subseteq TN$ is satisfied. This leads to a certain simplification and extension of the Marsden-Ratiu theorem.

Some examples

(a) $M = N = T^*G$; B tangent space to the orbits of the left action of G on T^*G. Then $\bar{N} = \mathfrak{g}^*$ with the Lie-Poisson left bracket.

(b) M Poisson manifold; $G \times M \to M$ left canonical action with equivariant momentum map J; μ regular value of J; $N = J^{-1}(\mu)$; G_μ isotropy subgroup; B tangent space to the G-orbits. Then $\bar{N} = J^{-1}(\mu)/G_\mu$ This example generalizes example (a).

(c) M as in example (b) $N = J^{-1}(\mathcal{O})$ where $\mathcal{O}$ is a coadjoint orbit; B tangent space to the G-orbits. Then $\bar{N}$ is diffeomorphic to N/G.

All these examples and many others appear in the article by Marsden and Ratiu

Generalization

In Cendra, H., Ratiu, T.S. and Yoshimura, H. (work on Dirac-Weinstein reduction, in progress) it has been proven that Marsden-Ratiu reduction is a case of Bivector reduction of Dirac vector bundles.