Una mirada sobre flujos cotangentes discretos no holónomos dependientes de un paso de tiempo

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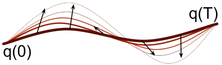
Resultados de nuestro interés en el caso sin vínculos

- (Q, L) sist. Lagrangiano regular
  - Q variedad diferencial de dimensión finita (*espacio de configuración*)
  - $L : TQ \to \mathbb{R}$ suave (*Lagrangiano*)
  - Trayectorias críticas de

$$S[\text{curva } q(t)] := \int_0^T L(q(t), \dot{q}(t))\, dt$$



$$\rightsquigarrow \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) = 0$$

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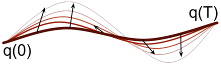

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Observaciones [PC09]

Discretización exacta: $\big(h, (q(0), \dot{q}(0))\big) \xmapsto{\partial^{\mp}} \big(h, (q(0), q(h))\big)$ difeomorfismo entre entornos $U - (\{0\} \times TQ)$ y $W - (\{0\} \times Q \times Q)$.

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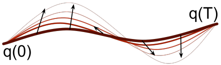

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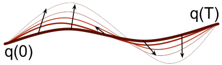

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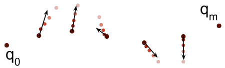

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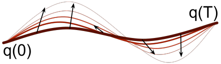

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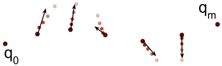

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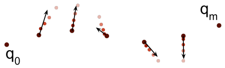

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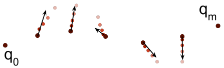

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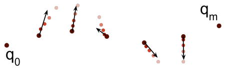

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$\tilde{F}_{L_{d,h}}$ (*flujo cotangente discreto*)

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TQ & & T^*Q \arrow[dd, "\tilde{F}_{L_{d,h}}"] \\
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& & T^*Q \\
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\end{tikzcd}$$

$\tilde{F}_{L_{d,h}}$ (*flujo cotangente discreto*)

Bondades de los sistemas mecánicos discretos [MW01] (y discretizados)

- Simplecticidad: $\tilde{F}^*_{L_{d,h}}\omega = \omega$.
- Teorema de Noether discreto.

Resultados de nuestro interés en el caso sin vínculos

- (Q, L) sist. Lagrangiano regular

$$\frac{\partial L}{\partial q}(q(t), \dot{q}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) = 0$$

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\end{tikzcd}$$

$\tilde{F}_{L_{d,h}}$ (*flujo cotangente discreto*)

Resultado clásico

$\frac{\partial}{\partial h}\tilde{F}_{L_{d,h}} = X^h \circ \tilde{F}_{L_{d,h}}$ es un campo Hamiltoniano dependiente del tiempo h (loc.).

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TQ \arrow[dr, "\partial_h^{\mp}"] & & T^*Q \\
 & Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] &
\end{tikzcd}$$

$\tilde F_{L_{d,h}}$ (*flujo cotangente discreto*)

Resultado clásico

$\frac{\partial}{\partial h}\tilde F_{L_{d,h}} = X^h \circ \tilde F_{L_{d,h}}$ es un campo Hamiltoniano dependiente del tiempo h (loc.).
Demostración (idea): $\frac{\partial}{\partial h}\tilde F^*_{L_{d,h}}\omega = \tilde F^*_{L_{d,h}}\left(i_{X^h}(d\omega) + d(i_{X^h}\omega)\right) = 0 \Rightarrow d(i_{X^h}\omega) = 0.$

Una posible construcción de flujos cotangentes discretos no holónomos

- $(Q, L, \mathcal{D})$ sist. Lagrangiano no holónomo
  - $L = \frac{1}{2}\langle \cdot, \cdot \rangle - V \circ \tau_Q$
  - $\mathcal{D} \subset TQ$ distribución de *vínculos cinemáticos* y *variacionales*
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$$
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TQ \arrow[dr, bend left, "\partial_h^{\mp}"] & & T^*Q \arrow[dd] & \mathcal{M} \\
& Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] \arrow[dr, "\mathbb{F}^+ L_{d,h}"] & & \\
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\mathcal{D} \arrow[r, hook] & TQ \arrow[r, "\mathbb{F}L"] & T^*Q \arrow[r, "p"] & \mathcal{M}
\end{tikzcd}
$$

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$$
\begin{tikzcd}
TQ \arrow[dr, bend left, "\partial_h^{\mp}"] & & T^*Q \arrow[dd] & \mathcal{M} \arrow[l, hook', "i"'] \arrow[dd, "\tilde{F}_{L_{d,h}}"] \\
& Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] \arrow[dr, "\mathbb{F}^+ L_{d,h}"] & & \\
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Teorema (existencia y convergencia)

Existe (loc.) $\tilde{F}_{L_{d,h}} := p \circ (\mathbb{F}^+ L_{d,h} \circ \partial_h^{\mp}) \circ (\mathbb{F}^- L_{d,h} \circ \partial_h^{\mp})^{-1}\big|_{\mathcal{M}} \to id_{\mathcal{M}}$ si $h \to 0$.

Relación con sistemas mecánicos discretos no holónomos

$$
\begin{tikzcd}
TQ \arrow[dr, bend left, "\partial_h^{\mp}"] & & T^*Q \arrow[dd] & \mathcal{M} \arrow[l, hook', "i"'] \arrow[dd, "\tilde{F}_{L_{d,h}}"] \\
& Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] \arrow[dr, "\mathbb{F}^+ L_{d,h}"] & & \\
& & T^*Q \arrow[r, "p"] & \mathcal{M} \simeq \mathcal{D}^* \\
& Q \times Q & &
\end{tikzcd}
$$

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& Q \times Q \arrow[ur, dashed, "\mathbb{F}^- L_{d,h}"] & &
\end{tikzcd}
$$

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$$
\begin{tikzcd}
TQ \arrow[drr, bend left, "\partial_h^{\mp}"] & & & T^*Q \arrow[dd] & \mathcal{M} \arrow[l, hook', "i"'] \arrow[dd, "\tilde{F}_{L_{d,h}}"] \\
& D_{K,h} \arrow[r, hook, "j_h"] \arrow[dd, "F_{L_{d,h}}"'] & Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] \arrow[dr, "\mathbb{F}^+ L_{d,h}"] \arrow[dd, dashed] & & \\
& & & T^*Q \arrow[r, "p"] & \mathcal{M} \simeq \mathcal{D}^* \\
& D_{K,h} \arrow[r, hook, "j_h"] & Q \times Q \arrow[ur, dashed, "\mathbb{F}^- L_{d,h}"] \arrow[urr, bend right, "p \circ \mathbb{F}^- L_{d,h}"'] & &
\end{tikzcd}
$$

Teorema (relación con SMDs no holónomos)

Existe (loc.) $F_{L_{d,h}}$ para la familia de *sistemas mecánicos discretos no holónomos* $(Q, L_{d,h}, D_{K,h}, \mathcal{D})$, donde $D_{K,h} := (\mathbb{F}^- L_{d,h})^{-1}(\mathcal{M}) \subset Q \times Q$ son los *vínculos cinemáticos discretos* y $\mathcal{D} \subset T^*Q$, los *vínculos variacionales discretos*.

Relación con sistemas mecánicos discretos no holónomos

$$
\begin{tikzcd}
TQ \arrow[drr, bend left=10, "\partial_h^{\mp}"] & & & T^*Q \arrow[dd] & \mathcal{M} \arrow[l, hook', "i"'] \arrow[dd, "\tilde{F}_{L_{d,h}}"] \\
& D_{K,h} \arrow[r, hook, "j_h"] \arrow[dd, "F_{L_{d,h}}"'] & Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] \arrow[dr, "\mathbb{F}^+ L_{d,h}"] \arrow[dd, dashed] & & \\
& & & T^*Q \arrow[r, "p"] & \mathcal{M} \simeq \mathcal{D}^* \\
& D_{K,h} \arrow[r, hook, "j_h"] & Q \times Q \arrow[ur, dashed, "\mathbb{F}^- L_{d,h}"] \arrow[urr, bend right=10, "p \circ \mathbb{F}^- L_{d,h}"'] & &
\end{tikzcd}
$$

Observación

De [BFG13], si $h \neq 0$ y $\omega_{L_{d,h}}^{D_{K,h}} := (\mathbb{F}^{\pm} L_{d,h} \circ j_h)^* \omega \in \Omega^2(D_{K,h})$, entonces

$$F_{L_{d,h}}^* \omega_{L_{d,h}}^{D_{K,h}} = \omega_{L_{d,h}}^{D_{K,h}} + d\xi_h,$$

con $\xi_h \in \Omega^1(D_{K,h})$ tal que

$$\xi_h(q_0, q_1)(\delta q_0, \delta q_1) := \left(\mathbb{F}^+ L_{d,h} - \mathbb{F}^- L_{d,h}(F_{L_{d,h}}(q_0, q_1))\right)(\delta q_1) \quad (= 0 \text{ si } \delta q_1 \in \mathcal{D}_{q_1}).$$

Relación con sistemas mecánicos discretos no holónomos

$$
\begin{tikzcd}
TQ \arrow[drr, bend left, "\partial_h^{\mp}"] & & & T^*Q \arrow[dd] & \mathcal{M} \arrow[l, hook', "i"'] \arrow[dd, "\tilde{F}_{L_{d,h}}"] \\
& D_{K,h} \arrow[r, hook, "j_h"] \arrow[dd, "F_{L_{d,h}}"'] & Q \times Q \arrow[ur, "\mathbb{F}^- L_{d,h}"] \arrow[dr, "\mathbb{F}^+ L_{d,h}"] \arrow[dd, dashed] & & \\
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Contraparte cotangente: $\tilde{F}_{L_{d,h}}^* \omega^{\mathcal{M}} = \omega^{\mathcal{M}} + d\beta_h,$

donde $\omega^{\mathcal{M}} := i^* \omega \in \Omega^2(\mathcal{M})$ y $\beta_h := (p \circ \mathbb{F}^- L_{d,h} \circ j_h)_* \xi_h \in \Omega^1(\mathcal{M})$.

¿Relación con campos Hamiltonianos no holónomos?

- $(Q, L, \mathcal{D})$ sist. Lagrangiano no holónomo
  - $\mathcal{C} := \{(\delta q, \delta p) \in TM : \delta q \in \mathcal{D}\}$
  - $i_{X_H}\omega^{\mathcal{M}}|_{\mathcal{C}} = dH^{\mathcal{M}}|_{\mathcal{C}}$; F_H^t flujo (*campo Hamiltoniano no holónomo*)

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[d, "F_L^t"] & TQ \arrow[r, "\mathbb{F}L"] & T^*Q \arrow[r, "p"] & \mathcal{M} \arrow[d, "\tilde{F}_L^t = F_H^t"] \\
\mathcal{D} \arrow[r, hook] & TQ \arrow[r, "\mathbb{F}L"] & T^*Q \arrow[r, "p"] & \mathcal{M}
\end{tikzcd}$$

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Problema abierto

¿$\frac{\partial}{\partial h}\tilde{F}_{L_{d,h}} = X^h \circ \tilde{F}_{L_{d,h}}$ cumple (loc.) $i_{X^h}\omega^{\mathcal{M}}|_{\mathcal{C}} = dH_h|_{\mathcal{C}}$ para $H_h : \mathcal{M} \to \mathbb{R}$?

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Dificultades esenciales

1. $X_h \subset \mathcal{C}$ si y sólo si $\frac{\partial}{\partial h}\left(p_2 \circ (\mathbb{F}^- L_{d,h})^{-1}|_{\mathcal{M}}\right)(q_0, p_0) \in \mathcal{D}$.

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Ejemplo: $(Q, L^E_{d,h} - W \circ p_2)$, con $L^E_{d,h}$ el *Lagrangiano discreto exacto* de un (Q, L) que preserva $\mathcal{D}$ (i.e. sin fuerzas de vínculo).

¿Relación con campos Hamiltonianos no holónomos?

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Dificultades esenciales

2. (*) vale si y sólo si $(\tilde{F}_{L_{d,h}})_*(\frac{\partial}{\partial h}\beta_h) \subset \mathcal{C}^\circ$.

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Dificultades esenciales

2. (*) vale si y sólo si $(\tilde{F}_{L_{d,h}})_*(\frac{\partial}{\partial h}\beta_h) \subset \mathcal{C}^\circ$.

Condición suficiente: $T\tilde{F}_{L_{d,h}}$ preserva $\mathcal{C}$.

¿Ejemplos dentro de 1 ?

Temas de investigación afines

- Actuales: reducción de simetrías y preservación de estructuras.

Temas de investigación afines

- Actuales: reducción de simetrías y preservación de estructuras.
- A futuro: orden de convergencia; backward error analysis.

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